ISRAEL JOURNAL OF MATHEMATICS 104 (1998), 51–124

A POSITIVE ANSWER TO THE BASIS PROBLEM

BY

PAOLO TERENZI

Dipartimento di Matematica del Politecnico
Piazza Leonardo da Vinci 32, 20133 Milano, Italy
e-mail: paoter@mate.polimi.it

ABSTRACT

$\{x_n\}$ with $\{x_n, x_n^*\}$ biorthogonal is a "uniformly minimal basis with quasi-fixed brackets and permutations" of a Banach space X if $\{x_n\}$ and $\{x_n^*\}$ are both bounded. Moreover, there is an increasing sequence $\{q_m\}$ of positive integers such that, for each x' of X, setting $q'(0) = 0$,

$$x' = \sum_{m=0}^{\infty} \sum_{n=q'(m)+1}^{q'(m+1)} x^*_{\pi'(n)}(x')x_{\pi'(n)},$$

where, for each $m \geq 1$, $q(m) + 1 \leq q'(m) \leq q(m+1)$ while

$$\{\pi'(n)\}_{n=q(m)+1}^{q(m+1)} \text{ is a permutation of } \{n\}_{n=q(m)+1}^{q(m+1)}.$$

Then, for each subspace Y of a separable Banach space X, there exists a uniformly minimal basis with quasi-fixed brackets and permutations of Y, which can be extended to a uniformly minimal basis with quasi-fixed brackets and permutations of X.

TABLE OF CONTENTS

Received March 4, 1993 and in final revised form October 8, 1996

Introduction

The main idea was to equip a normed space, in particular an infinite-dimensional separable Banach space, by means of a system of coordinate axes, like the couple of abscissa axis and ordinate axis which characterize R^2. From this idea a famous problem was born and for a long time remained unsolved: The **basis problem**. Indeed for a Hilbert space this goal was achieved by any orthonormal basis, while for general Banach spaces the situation was quite different. The search for a suitable system of coordinate axes originated already in Banach's book [4] with the problem of the existence of the basis. This problem was solved only in 1973 by Enflo in the negative. At this point let us give a list of four main ideas (definitions $D_1, \ldots, D_4$) of systems of coordinate axes in a general separable infinite-dimensional Banach space. We shall begin with the more general (hence less useful, but which exists in every case) definition and proceed with more and more restrictive conditions (which are satisfied only in particular cases).

If X is a Banach space, X^* its dual and if $\{x_n\}$ is a *minimal* sequence of X (that is, $\text{dist}(x_m, \text{span}\{x_n\}_{n \neq m}) > 0$ for each m, hence there exists $\{x_n^*\}$ in X^* with

$$\{x_n, x_n^*\} \text{ biorthogonal},$$

that is, $x_m^*(x_n) = \delta_{mn}$ Kronecker indices), then we can consider $\{x_n\}$ as a system of coordinate axes and, in particular, we call this system:

(D_1) **M-basis** (from *Markushevich basis*) if $\text{span}\{x_n\}$ is dense in X and $\{x_n^*\}$ is **total** on X ($x_n^*(x) = 0$ for each n imply $x = 0$), hence each element x of X is associated in a unique way with the series $\Sigma_{n=1}^{\infty} x_n^*(x) x_n$ without any information on the convergence; in particular the M-basis is said to be **norming** if there exists $K > 0$ such that, for each x of X,

$$\|x\| \leq K \cdot \sup\{|x^*(x)| : x^* \in \text{span}\{x_n^*\} \text{ and } \|x^*\| = 1\};$$

moreover, the M-basis is said to be **strong** if $x \in$ closure of $\text{span}\{x_n^*(x) x_n\}$ for each x of X (in general, an M-basis is neither norming nor strong).

(D_2) **Uniformly minimal M-basis** if in the preceding definition $\{x_n\}$ and $\{x_n^*\}$ are both bounded (hence there is a $K > 0$ so that $\text{dist}(x_m, \text{span}\{x_n\}_{n \neq m}) \geq K$ for each m), this implies for the preceding series that

$$x_n^*(x) x_n \to 0 \quad \text{with } n,$$

which is a necessary but not sufficient condition for the convergence (let us point out this simple but important fact (see Reference I* at the end of this

Introduction): if $\|x_n\| = 1$ and $\|x_n^*\| < H$ for each n, if $x \in X$ and $\varepsilon > 0$, there exist

$$w' = \sum_{n=1}^{p} x_n^*(x)x_n + \sum_{n=p+1}^{p+q} a_n x_n \quad \text{and} \quad w'' = \sum_{n=1}^{p+q+r} x_n^*(x)x_n + \sum_{n=p+q+r+1}^{p+q+r+s} a_n x_n$$

with $\|x - w'\| < \varepsilon/2H$ and $\|x - w''\| < e/2H$, hence if

$$w = w'' - w' = \sum_{n=p+1}^{p+q} [x_n^*(x) - a_n]x_n + \sum_{n=p+q+1}^{p+q+r} x_n^*(x)x_n + \sum_{n=p+q+r+1}^{p+q+r+s} a_n x_n$$

we have that $\|w\|$ $(= \|(x - w') - (x - w'')\|) < \varepsilon/H$; on the other hand, for $p+q+1 \leq n \leq p+q+r$, $x_n^*(x) = x_n^*(w)$, hence

$$|x_n^*(x)| \leq \|x_n^*\| \cdot \|w\| < \varepsilon.$$

(D_3) **Basis with individual brackets and permutations** if for each x' of X there exist an increasing sequence $\{q'(m)\}$ of positive integers and a permutation $\{\pi'(n)\}$ of the sequence $\{n\}$ of the natural numbers such that, setting $q'(0) = 0$,

$$x' = \sum_{m=0}^{\infty} \sum_{n=q'(m)+1}^{q'(m+1)} x^*_{\pi'(n)}(x')x_{\pi'(n)}; \tag{1}$$

in particular, if we have always $q'(m) = m$ for each $m \geq 0$, we call $\{x_n\}$ a **basis with individual permutations**; while if we have always $\pi'(n) = n$ for each n, we call it a **basis with individual brackets**.

(D_4) **Uniformly minimal basis with quasi-fixed brackets and permutations** if (D_2) and (D_3) hold, moreover there exists a fixed increasing sequence $\{q(m)\}$ of positive integers such that for each x' of X we have (1), where the sequence $\{q'(m)\}$, though not fixed, is not entirely arbitrary: it is regulated by $\{q(m)\}$ in the sense that each $q'(m)$ can only range on a finite interval specified as follows:

$$q(m) + 1 \leq q'(m) \leq q(m+1) \quad \text{for each } m \geq 1. \tag{2}$$

Moreover, $\{\pi'(n)\}$ is also not quite arbitrary: it is the union of finite permutations regulated, again by $\{q(m)\}$, in the following way. Setting $q(0) = 0$, for each $m \geq 0$

$$\{\pi'(n)\}_{n=q(m)+1}^{q(m+1)} \text{ is a permutation of } \{n\}_{n=q(m)+1}^{q(m+1)}. \tag{3}$$

In particular, we say that $\{x_n\}$ is a **uniformly minimal basis with quasi-fixed brackets** if in (3) we have always $\pi'(n) = n$ for each n.

Finally, we say that $\{x_n\}$ is a **uniformly minimal basis with fixed brackets** if it is a uniformly minimal basis with quasi-fixed brackets, such that in (2) we have always $q'(m) = q(m)$ for each $m \geq 0$.

Clearly $\{x_n\}$ is a **basis** in the usual sense if it is a uniformly minimal basis with fixed brackets such that in (2) we have always $q'(m) = q(m) = m$ for each m. We point out that by **basis with fixed brackets** (better known as **basis with brackets**) we mean the same definition of uniformly minimal basis with fixed brackets, but without (D_2). In general a basis with fixed brackets need not be uniformly minimal.

Regarding the questions of existence we present two theorems:

THEOREM I: *Every separable Banach space has a uniformly minimal basis with quasi-fixed brackets and permutations.*

In the research on Banach spaces sometimes the following situation occurs: we have a subspace Y of a separable Banach space X and we need some kind of basis of Y (with the best possible properties) which can be extended to a basis of the same kind of X. Let us point out that, even in the particular case where X and Y have bases, in general [48] Y need not have a basis which can be extended to a basis of X. Extending bases is possible in certain cases, for example [63] if Y has a basis which is a block sequence of a basis of X (we also recall [58] that, if X and Y have a basis, there always exists a subsequence of this basis which can be extended to a basis of X). The same is true for the bases with brackets (further information about properties of extension of weaker kinds of bases can be found in References at the end of this Introduction). Let us present the following property of extension, which improves Theorem I:

THEOREM II: *Every subspace of a separable Banach space X has a uniformly minimal basis with quasi-fixed brackets and permutations, which can be extended to a uniformly minimal basis with quasi-fixed brackets and permutations of X.*

We outline the organization of the paper. In §1 and §2 we set up all the tools we shall use in the proof of Theorem I, which is in §3. We point out that in a preceding unpublished version, in a more axiomatic way and practically mainly by means of the definition of norm, we proved directly Theorem I without passing through the properties of §1 on finite-dimensional classical Banach spaces; but the central part of that version based itself on a heavy combinatorial technique which more than doubled the length of the proof.

§1 is informal and the main fact discussed there is the simple idea of the finite transformability of Banach spaces; after that we only present refinements of known facts. It is known that a Banach space Y is "finitely represented" in a Banach space X if, for each finite-dimensional subspace Y_0 of Y and for each $\varepsilon > 0$, there exists a subspace X_0 of X which is $(1+\varepsilon)$-isomorphic to Y_0. Proceeding from another point of view, in this Note we say that X is "finitely transformable" in Y if, for each finite-dimensional subspace Y_0 of Y and for each $\varepsilon > 0$, there exists a subspace X_0 of X such that X/X_0 is $(1+\varepsilon)$-isomorphic to Y_0: That is, by means of division by the subspace X_0, X can be "transformed" into a space $(1+\varepsilon)$-isomorphic to Y_0. An advantage of this new definition is the following fact (proved in §1): The finite representability of l_1 in X, and the finite transformability of X in c_0, are the same thing.

The strategy of the first step of the proof is to consider separately two cases: If the space has type > 1 we already have at our disposal a sequence of uniformly complemented copies of l_2^n (see Reference IV* at the end of this Introduction), while if the space does not have type > 1, by means of the properties of §1 we shall be able to construct a sequence $\{V_m\}$ of subspaces of X of the following kind:

(a) There exists a biorthogonal system $\{\{v_{m,n}, v^*_{m,n}\}_{n=1}^{r(m)}\}_{m=1}^{\infty}$ such that, for each m, $X = V_m + V_{0,m}$ with $V_m = \operatorname{span}\{v_{m,n}\}_{n=1}^{r(m)}$,

$$V_{0,m} = X \cap \{\bigcap_{n=1}^{r(m)} v^*_{m,n\perp}\} \text{ and } \{v_{m,n} + V_{0,m}\}_{n=1}^{r(m)} \text{ is} \tag{4}$$

H_m-equivalent to the natural basis of $l_1^{r(m)}$, with $r(m)/H_m > 2^m$;

(b) $\|v_{m,n}\| = 1$ and $\|v^*_{m,n}\| < K$ for $1 \le n \le r(m)$ and for each m (from (a) it directly follows that, for each x_0 of X and for each m, there exists $n(m) = n(m, x_0)$ with $1 \le n(m) \le r(m)$ such that $|v^*_{m,n(m)}(x_0)| \le \|x_0\|/2^m$; indeed as indicated above

$$\|x_0\| \ \ge\ \|x_0 + V_{0,m}\| = \left\| \sum_{n=1}^{r(m)} v^*_{m,n}(x_0)v_{m,n} + V_{0,m}\right\| \ \ge\ \frac{1}{H_m}\sum_{n=1}^{r(m)} |v^*_{m,n}(x_0)|$$
$$\ge \frac{r(m)}{H_m}\min\{|v^*_{m,n}(x_0)| : 1 \le n \le r(m)\};$$

it also follows that

$$H_m \left\|\sum_{n=1}^{r(m)} v^*_{m,n}(x_0)v_{m,n} + V_{0,m}\right\| (\ge \sum_{n=1}^{r(m)} |v^*_{m,n}(x_0)|) \ge \left\|\sum_{n=1}^{r(m)} v^*_{m,n}(x_0)v_{m,n}\right\|,$$

hence there exists a projection $P_m : X \to V_m$ with $V_{0,m} = X \cap P_{m\perp}$ and $\|P_m\| \leq H_m$).

The heart of the proof of Theorem I is contained in §2 and §3, where we only use the technique of the biorthogonal systems. In §2 we obtain (4) by means of the following procedure: Since the space does not have type > 1, l_1 is finitely represented in X, hence X is finitely transformable in c_0; at this point we use the fact that, for each m, l_∞^m has a biorthogonal system $\{x_n, x_n^*\}_{n=1}^m$ such that $\{x_n\}_{n=1}^m$ is H_m-equivalent to the natural basis of l_1^m, with $m/H_m \to +\infty$ (in order to have (a) of (4)), moreover with $\|x_n\| = 1 = \|x_n^*\|$ for $1 \leq n \leq m$ (in order to have (b) of (4)).

Definition: A uniformly minimal norming M-basis $\{x_n\}$ is said to have **controlled coefficients** if, for each sequence $\{\varepsilon_m\}$ of positive numbers with $\varepsilon_m \to 0, \{x_n\}$ of (D_2) has a fixed partition in blocks $\{\{x_n\}_{n=q(m)+1}^{q(m+1)}\}_{m=1}^\infty$ such that, for each x of X with $\|x\| = 1$ and for each m, there exists n with $q(m)+1 \leq n \leq q(m+1)$ and $|x_n^*(x)| < \varepsilon_m$. Hence we have a kind of control of the coefficients.

In the second step of the proof in §2 we use the first step to construct in any separable Banach space a uniformly minimal norming M-basis with controlled coefficients. We point out that a basis (for instance, the natural basis of c_0) does not have in general controlled coefficients.

The last step is in §3 where we pass, from the uniformly minimal norming M-basis $\{x_n\}$ with controlled coefficients, to $\{y_n\}$, a uniformly minimal basis with quasi-fixed brackets and permutations of X, by means of a **block perturbation** (in the following sense: There is an increasing sequence $\{q(m)\}$ of positive integers such that, for each m,

$$\operatorname{span}\{y_n\}_{n=q(m)+1}^{q(m+1)} = \operatorname{span}\{x_n\}_{n=q(m)+1}^{q(m+1)}).$$

We wish to point out in advance the main idea of this paragraph (disregarding uniform minimality): If $\|x_n\| = 1$ for each n, there exists an increasing sequence $\{r_m\}$ of positive integers such that, if x_0 is an element of X, we can find a subsequence $\{s(m)\}$ of $\{m\}$ and, for each m, a suitable $n(m)$ with $r_{s(m)+1} + 1 \leq n(m) \leq r_{s(m+1)}$, such that

$$\left\| x_0 - \left\{ \sum_{n=1}^{s(m)} y_n^*(x_0) y_n + y_{n(m)}^*(x_0) y_{n(m)} \right\} \right\| < \frac{1}{2^m}$$

(indeed at first we can choose $\{s(m)\}$ of $\{m\}$ such that, for each m, there exists $v_m \in \operatorname{span}\{x_n\}_{n=r_{s(m)}+1}^{r_{s(m)+1}}$ such that

$$\Big\|x_0 - \Big\{\sum_{n=1}^{r_{s(m)}} x_n^*(x_0)x_n + v_m\Big\}\Big\| = \Big\|x_0 - \Big\{\sum_{n=1}^{r_{s(m)}} y_n^*(x_0)y_n + v_m\Big\}\Big\| < \frac{1}{2^{m+1}};$$

moreover the construction of the block perturbation allows the possibility to choose an index $n(m)$ such that we can assume that we have the following situation:

$$y_{n(m)}^* = x_{n(m)}^* + w_m^* \quad \text{and} \quad y_{n(m)} = x_{n(m)} + \frac{v_m}{w_m^*(x_0)},$$

with $|w_m^*(x_0)| < 1/2^{m+2}$ and (by the controlled coefficients property) $|x_{n(m)}^*(x_0)| < 1/2^{m+2}\|y_{n(m)}\|$; hence

$$\begin{aligned}
&\Big\|x_0 - \Big\{\sum_{n=1}^{r_{s(m)}} y_n^*(x_0)y_n + y_{n(m)}^*(x_0)y_{n(m)}\Big\}\Big\| \\
&\le \Big\|x_0 - \Big\{\sum_{n=1}^{r_{s(m)}} x_n^*(x_0)x_n + v_m\Big\}\Big\| + \|v_m - y_{n(m)}^*(x_0)y_{n(m)}\| \\
&< \frac{1}{2^{m+1}} + \|v_m - y_{n(m)}^*(x_0)y_{n(m)}\| \\
&= \frac{1}{2^{m+1}} + \Big\|v_m - \{x_{n(m)}^*(x_0) + w_m^*(x_0)\}\Big\{x_{n(m)} + \frac{v_m}{w_m^*(x_0)}\Big\}\Big\| \\
&= \frac{1}{2^{m+1}} + \|w_m^*(x_0)x_{n(m)} + x_{n(m)}^*(x_0)y_{n(m)}\| \\
&\le \frac{1}{2^{m+1}} + \|w_m^*(x_0)x_{n(m)}\| + \|x_{n(m)}^*(x_0)y_{n(m)}\| \\
&< \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \frac{1}{2^{m+2}} = \frac{1}{2^m}).
\end{aligned}$$

§4 contains the main part of the proof of Theorem II: The extension of the M-basis with controlled coefficients and some new devices on the biorthogonal systems, while the actual proof of Theorem II is in §5 and the construction is analogous to the construction of §3.

Now some comments.

Comment 1: From a historical point of view: For many years the best known positive answer to the basis problem was the existence of the uniformly minimal M-basis (Ovsepian–Pelczyński [32]), which answered positively a question that again originated in Banach's book (with a different terminology). In 1976

Pelczyński [33] refined this result by proving the existence of an M-basis $\{x_n\}$ with $\{x_n, x_n^*\}$ biorthogonal and $\|x_n\| \cdot \|x_n^*\| < 1 + \varepsilon$ for each n, for each fixed $\varepsilon > 0$ (if $\varepsilon = 0$ it is called an **Auerbach biorthogonal system** or an **Auerbach M-basis** and the question of its existence in every separable Banach space is still open, though a negative answer is expected). Only recently [57] we improved the positive answer of [32] by means of the existence of the strong M-basis (the question of the existence of the strong M-basis has been open since 1970). However, it is evident that also this improvement was quite far from the uniformly minimal basis with quasi-fixed brackets and permutations of Theorem I.

Regarding the negative answers we only recall that the basis with brackets ("basis with parenthesis" in [49] p. 450) corresponds to the *finite-dimensional Schauder decomposition* of the space ([25] p. 48), hence already this kind of basis in general does not exist (Enflo [6], otherwise the space would have the approximation property); we also recall [51] that there are spaces with the approximation property but without a basis.

Therefore the couple of answers, positive for the uniformly minimal basis with quasi-fixed brackets and permutations and negative for the basis with brackets, is a sufficiently good approximation of the border between the kinds of bases which exist in every separable Banach space and the kinds of bases which exist only in particular cases; this border was the goal of the "basis problem", hence the end of its history.

Comment 2: On the basis with individual brackets and permutations: We recall (Revesz [44]) that the trigonometric system is a basis with individual brackets and permutations of the space of the continuous 2π-periodical functions (but not a basis with brackets, by a counterexample of Du Bois-Reymond). We point out that the basis with quasi-fixed brackets and permutations is strictly stronger than the basis with individual brackets and permutations: For instance, the first one is always norming ([55] §2, Proposition 1), while the second one is not in general norming ([53] §2, example).

Comment 3: On the (uniformly minimal) basis with quasi-fixed brackets: The question of the existence is still open only in the spaces of finite cotype; indeed in [55] (Theorem I) we proved, in the spaces where c_0 is finitely represented, the existence of a basis with quasi-fixed brackets; moreover, by means of the ideas of §1–§3, it is possible to improve the situation in these spaces and to obtain the uniformly minimal basis with quasi-fixed brackets. Since there are subspaces of c_0 without the approximation property (but with a uniformly minimal basis with quasi-fixed brackets as indicated above), it follows that the (uniformly minimal) basis with brackets is strictly stronger than the (uniformly minimal) basis with quasi-fixed brackets.

Comment 4: On the basis with individual permutations: At first we recall [19] that the basis is strictly stronger than the basis with individual permutations. Although the question of the existence is still open, in [57] we already solved in the positive the weak version of this question, i.e. the question (raised in [16]) on the existence of the **Steinitz basis** (that is, an M-basis $\{x_n\}$ of X and with $\{x_n, x_n^*\}$ biorthogonal, such that, for each x of X and for each x^* of X^*, there exists a permutation $\{\pi(n)\}$ of $\{n\}$ such that $x^*(x) = \sum_{n=1}^{\infty} x_{\pi(n)}^*(x)x^*(x_{\pi(n)})$). However, also the uniformly minimal basis with quasi-fixed brackets and permutation of Theorem I is a Steinitz basis, since ([16] Prop. 5, p. 86) "Every uniformly minimal basis with individual brackets and permutations is a Steinitz basis".

Comment 5: On other results: Kadets, Plichko and Popov [20] introduced, in a Banach space X, the notion of "*finite basis*" (that is, a sequence $\{x_n\}$ with span$\{x_n\}$ dense in X and with a number $K > 0$ such that every finite subsequence has a permutation with basis constant $\leq K$). They gave examples of Banach spaces X with finite bases which were not rearranged bases; they also proved that in these cases X can be decomposed into a direct sum of two infinite-dimensional subspaces; in particular the finite basis does not exist in every separable X.

We recall [51] that in general a separable Banach space does not have the *local basis structure* (that is, a sequence of finite-dimensional subspaces $\{E_n\}$ with $\{\bigcup_1^{\infty} E_n\}$ dense in X, such that, for each n, $E_{n+1} \supset E_n$ and $\mathrm{bc}(E_n) \leq C$, where

$$\mathrm{bc}(E_n) = \inf\{\text{basis constant of a basis of } E_n\}).$$

Finally, for other kinds of bases, we recall for instance [62] that every separable Banach space X containing c_0 has a RUC ("random unconditional convergence") system (that is, (D_1) such that, for every x in X, the series $\sum_{n=1}^{\infty} r_n(\omega)x_n^*(x)x_n$ converges almost surely in ω, where $\{r_n\}$ are the Rademacher functions).

We pass to some known results.

References I* ... V* below are properties which we shall often use in our proofs. Although we shall not use References VI* ... VIII* (since they concern properties of extension which are stronger than the property of extension of Theorem II), they give a panorama of known properties of extension of weaker kinds of bases from a subspace to the whole space. Finally, it seems useful to us to end this Introduction with a short outline of the "basis problem" for the non-separable case, always from the point of view of the M-bases.

I* (see Theorem I and proof of lemma of [53], see also II* of §1 of [57]; it can also be derived either from the results of [18] or from [17] (Theorem 3)).

"If $\{x_n, x_n^*\}$ is biorthogonal with $\text{span}\{x_n\}$ dense in X, there exists an increasing sequence $\{r_m\}$ of positive integers such that, for each m and for each x of $\text{span}\{x_n\}_{n=1}^{r_m}$,

$$|\operatorname{dist}(x, \text{span}\{x_n\}_{n>r_m}) - \operatorname{dist}(x, \text{span}\{x_n\}_{n=r_m+1}^{r_{m+1}})| \leq \|x\| / \Big\{2^m \Big(2 + \sum_{n=1}^{r_m} \|x_n^*\| \cdot x_n\|\Big)\Big\};$$

then it follows that, for each x' of X,

$$x' = \lim_{m\to\infty} \Big\{ \sum_{n=1}^{r_m} x_n^*(x')x_n + \sum_{n=r_m+1}^{r_{m+1}} a_n' x_n \Big\}$$

where $\{a_n'\}$ depends on x' while $\{r_m\}$ does not depend on x'.

Moreover, if $\{x_n\}$ is a norming M-basis of X, assuming the existence of an infinite subsequence $\{m(k)\}$ of $\{m\}$ such that, for each k,

$$\Big\| \sum_{n=r_{m(k)}+1}^{r_{m(k)+1}} x_n^*(x')x_n \Big\| < \varepsilon_k, \qquad \text{with } \sum_{k=1}^{\infty} \varepsilon_k \text{ finite,}$$

setting $r_{m(0)} = 0$, we have that $x' = \sum_{k=0}^{\infty} \sum_{n=r_{m(k)}+1}^{r_{m(k+1)}} x_n^*(x')x_n$."

We point out that M. I. Kadets [18] proved even more: There exist continuous functions f_n, depending on r_m variables, such that we may take

$$a_n' = f_n(x_1^*(x'), \ldots, x_{r_m}^*(x')).$$

V. P. Fonf [7] proves that every M-basis with this last property is norming; this result has been simplified and generalized by M. I. Ostrovskii [31].

II* ([32], see also [49] p. 248). *Let* $\{x_n, x_n^*\}_{n=1}^{2^Q}$ *be a biorthogonal system of* X*; there exists another biorthogonal system* $\{y_n, y_n^*\}_{n=1}^{2^Q}$ *with* $\text{span}\{y_n\}_{n=1}^{2^Q} = \text{span}\{x_n\}_{n=1}^{2^Q}$ *and* $\text{span}\{y_n^*\}_{n=1}^{2^Q} = \text{span}\{x_n^*\}_{n=1}^{2^Q}$ *such that, for every* n *with* $1 \leq n \leq 2^Q$,

$$\|y_n\| < \|x_1\|/2^{Q/2} + (1 + 2^{1/2}) \max\{\|x_k\|; 2 \leq k \leq 2^Q\}$$

and

$$\|y_n^*\| < \|x_1^*\|/2^{Q/2} + (1 + 2^{1/2}) \max\{\|x_k^*\|; 2 \leq k \leq 2^Q\}.$$

III* ([22], see also [25] (lemma 2.c.8 p. 77) or [47] p. 269). *If U and V are finite-dimensional subspaces of a normed space with dimension of $V >$ dimension of U, V has an element $x \neq 0$ with* $\operatorname{dist}(x, U) = \|x\|$.

IV* ([34] coroll. 2.12, see also [28] p. 112). *Let X be an infinite-dimensional Banach space with type $p > 1$. Then there exist a finite positive number C and, for each $\varepsilon > 0$ and positive integer m, an integer $N = N(\varepsilon, m)$ such that, if Y is a subspace of X with dimension $\geq N$, then Y contains an m-dimensional subspace Y_0 with $d(Y_0, l_2^m) \leq 1+\varepsilon$ and there exists a projection of norm $\leq C$ from X onto Y_0.*

Hence by [27], if l_1 is not finitely represented in X, we have the existence of a uniformly complemented sequence of copies of l_2^n (where a sequence $\{V_m\}$ of subspaces of X is **uniformly complemented** in X if there exists a sequence $\{P_m\}$ of projections such that $P_m : X \to V_m$ with $\|P_m\| < K$ for each m, where K does not depend on m); but the weaker condition "c_0 is not finitely represented in X" is not sufficient to imply this property [35].

V* ([4] (without proof), see also [25] Prop.1.c.3 p. 16 or [45] Th. 2. p. 257). *Every m-dimensional Banach space with m finite has a basis of Auerbach (that is,*

$$\{x_n, x_n^*\}_{n=1}^m \textit{biorthogonal with } \|x_n\| = \|x_n^*\| = 1 \textit{ for } 1 \leq n \leq m).$$

Now we pass to properties of extension of M-basic sequences and of construction of M-bases "in given directions", that is, through given subspaces. In what follows X is a separable Banach space, Y and Z are two subspaces of X with $Y \cap Z = \{0\}$ and $Y+Z$ dense in X; we say that Y and Z are **quasi-complements** of each other.

VI* ([54] Prop. 3, p. 502). *If $\{y_n\}$ is a norming M-basis of Y, there exists $\{z_n\}$ in Z such that $\{y_n\} \cup \{z_n\}$ is a norming M-basis of X.*

It is not possible to improve VI* by requiring $\{z_n\}$ to be an M-basis of Z, even if we do not require it to be norming ([46] Corollary 3, p. 186); but improvement becomes possible under stronger conditions, indeed: ([54] Prop. 4, p. 505). "Let $\{y_n\}$ be a norming M-basis of Y, then:

There exists an M-basis $\{z_n\}$ of Z with $\{y_n\} \cup \{z_n\}$ a norming M-basis of X $\Leftrightarrow$ There exists $\{y_n^*\}$ of $Z_\perp$ such that $\{y_n, y_n^*\}$ is biorthogonal".

VII* ([54] Th. II p. 498). *If $\{y_n\}$ is a uniformly minimal norming M-basis of Y, there exists $\{z_n\}$ in X such that $\{y_n\} \cup \{z_n\}$ is a uniformly minimal norming M-basis of X.*

Again it is not possible to improve VII* by requiring $Z \supset \{z_n\}$ ([52] Example).

We pass to the construction of the M-basis in the two directions Y and Z.

VIII* ([54] Th. III, p. 498). *There exists a uniformly minimal norming M-basis* $\{y_n\} \cup \{z_n\}$ *of* X *with* $Y \supset \{y_n\}$ and $Z \supset \{z_n\}$.

Again, it is not possible to improve VIII* by requiring $\{y_n\}$ to be an M-basis of Y, even if we do not require it to be norming ([52] Example).

Finally, about the extension of basic sequences, we point out also that, *if* $\{y_n\}$ *is basic* (that is, a basis of its closed linear span) *with* $\|y_n\| = 1$ *for each* n, there are always $\{f_n\}$ and $\{g_n\}$ basic, but [56] (Th. p. 189) it is not possible to always obtain these two properties simultaneously, that is, *in general there does not exist a bounded basic* $\{y_n^*\}$ *of* X *such that* $\{y_n, y_n^*\}$ *is biorthogonal.*

We pass to consider the "basis problem" for the non-separable case (an exhaustive outline of this subject up to 1980 can be found in [49], §17–§19).

At first we recall that the definition of M-basis for a *non-separable Banach space* X is again (D_1), only now we have, instead of the countable sequence $\{x_n, x_n^*\}$, the transfinite sequence $\{x_i, x_i^*)_{i\in I}$ with l non-countable. We begin to point out that in general X does not have an M-basis: Already l_I^∞ with l non-countable does not have an M-base (Dyer [5]); what's more (Plichko [39]) *in general* X *does not have a fundamental biorthogonal system* $\{x_i, x_i^*\}_{i\in I}$ (where *fundamental* means that $\text{span}\{x_i\}_{i\in I}$ is dense in X); in particular the non-separable Banach space $C(k)$, where K is Kunen's compact, does not have uncountable biorthogonal systems (Negrepontis [30]).

If X has a fundamental biorthogonal system, it is always possible to make it uniformly minimal, precisely with $\sup\{\|x_i\| \cdot \|x_i^*\| : i \in I\} < 4 + \varepsilon$, for each $\varepsilon > 0$ (Plichko [40]; a weaker version is that of Godun [10]). But, *if* X *is weakly compactly generated*" (there is a weakly compact subset C which generates X), then X has an M-basis (Lindenstrauss [24]); and X *has a uniformly minimal M-basis* (Plichko [36] and [37]). Moreover, if we do not require that $\text{span}\{x_i\}_{i\in I}$ be dense in X, the answer becomes positive, indeed (Plichko [38]):

IX*. *Every Banach space* X *has a bounded biorthogonal system* $\{x_i, x_i^*\}_{i\in I}$ *with* $\{x_i^*\}_{i\in I}$ *total on* X.

Regarding the quasi-complements: For the non-separable case it is not known if there exists an infinite-dimension separable subspace with a quasi-complement. However ([54] Prop. 2, p. 502) if $\{y_n\}$ is a uniformly minimal sequence of unit norm vectors of a non-separable X, in general there is no bounded $\{y_n^*\}$ in X^*, with $\{y_n, y_n^*\}$ biorthogonal, such that $\text{span}\{y_n\} + (\text{span}\{y_n^*\})_\perp$ is dense in X. Moreover [11] in l_∞ there are two quasi-complemented subspaces Y and Z such that, if $\{y_i\}$ and $\{z_i\}$ are minimal systems with $Y \supset \{y_i\}$ and $Z \supset \{z_i\}$, $\text{span}\{y_i\}$

dense in Y and span$\{z_i\}$ dense in Z, then $\{y_i\} \cup \{z_i\}$ is never minimal.

Regarding the existence of an M-basis, we recall that this fact implies that X has an equivalent norm such that in the new norm X shares many properties of l_1 (I) [13]).

Regarding the strong M-bases: The existence of the M-basis does not imply the existence of the strong M-basis [2]; while the existence of the strong M-basis implies the existence of an equivalent locally uniformly convex norm [1]; on the other hand, there is in [41] an example of a Banach space with an equivalent locally uniformly convex norm, but without M-bases.

Recently, we have been informed about the following results of Aleksandrov and Plichko [3], about the relations between the strong M-basis and the norming M-basis, for a Banach space X:

(i) If X has a norming (countable norming) M-basis, then X has a strong M-basis.

(ii) There exists a Banach space X with a strong M-basis, but without a norming M-basis.

Finally, about the Auerbach M-bases we recall [12], [14] and [43].

ACKNOWLEDGEMENT: To Professor Zippin, who checked the paper and improved the English and the terminology; in particular for his encouragement.

1. The finite transformability of Banach spaces

Two well-known definitions: We say that a sequence $\{y_n\}_{n=1}^m$ $(1 \le m \le \infty)$ is $(1+\varepsilon)$-**equivalent** to another sequence $\{x_n\}_{n=1}^m$ of X if, for any $\{a_n\}_{n=1}^p$ of numbers ($p = m$ if m is finite, otherwise $1 \le p < \infty$),

$$\frac{1}{1+\varepsilon}\Big\|\sum_{n=1}^{p} a_n x_n\Big\| \le \Big\|\sum_{n=1}^{p} a_n y_n\Big\| \le (1+\varepsilon)\Big\|\sum_{n=1}^{p} a_n x_n\Big\|.$$

We recall that the **projection constant** $\lambda(X)$ of a finite-dimensional Banach space X is $\min\{\lambda$: for all $Y \supset X$ there is a projection of norm $\le \lambda$ from Y on $X\}$.

Now three further definitions. If X and Y are two Banach spaces, we say that:

(D_5) $\qquad Y$ is finitely constructible in X

if, for any $\varepsilon > 0$ and for any finite-dimensional subspace Y_0 of Y, there is some couple of subspaces X_0 and Z_0 of X such that $d(Y_0, X_0/Z_0)$ (the **Banach–Mazur distance**) $< 1+\varepsilon$.

Assume that Y is finitely constructible in X; we say in particular that:

(D$_6$) Y is finitely represented in X

if in (D$_5$) we can always have $Z_0 = \{0\}$;

(D$_7$) X is finitely transformable in Y

if in (D$_5$) we can always have $X_0 = X$.

(D$_6$) is already well-known in the literature. Then (D$_6$) and (D$_7$) become two special cases, in opposite directions, of (D$_5$). In this paragraph we are mainly concerned with (D$_6$) and (D$_7$); indeed (D$_5$) will appear only in Proposition 1.3.

The next two propositions are already partially known and they connect (D$_6$) and (D$_7$), in the particular cases of $Y = c_0$ and of $Y = l_1$.

The main fact of the first proposition is the characterization of the finite representability of l_1 in X, by means of the finite transformability of X in c_0; another fact is that in general there are no relations between (D$_6$) and (D$_7$): Indeed, for c_0, (D$_6$) is strictly stronger than (D$_7$), while for l_1 it is the contrary.

PROPOSITION 1.1: *For a Banach space X we have the following implications and equivalences:*

(a) *c_0 is finitely represented and uniformly complemented in X, with the projections bounded in norm by $1 + \varepsilon$, for each fixed $\varepsilon > 0$.*

$\Leftrightarrow$(b) *c_0 is finitely represented in X.*

$\Leftrightarrow$(c) *X is finitely transformable in c_0.*

$\Leftrightarrow$(d) *l_1 is finitely represented in X.*

$\Leftrightarrow$(e) *X is finitely transformable in l_1.*

$\Leftrightarrow$(f) *l_1 is finitely represented and uniformly complemented in X, with the projections bounded in norm by $1 + \varepsilon$, for each fixed $\varepsilon > 0$.*

Remark 1.1: The implication (b) $\Rightarrow$ (a) is well-known, since $\lambda(l_\infty^n) = 1$ for each n; in particular the spaces l_∞^n are the only finite-dimensional spaces whose projection constant is 1 ([9] and [29]; moreover see [65] for a very simple direct proof). We also recall that (b) $\Rightarrow$ (a) is the finite version of the well-known fact (Sobczyk [50], for a shorter proof see Veech [59] (also [25] Th. 2.f.5, p. 106; or [61] Th. 4, p. 146), for another simple proof by means of the biorthogonal systems [54] (Remark, p. 502)):

"If c_0 is a subspace of a separable X, then c_0 is complemented in X and there exists a projection $P : X \to c_0$ with $\|P\| \leq 2$"; moreover, this fact characterizes c_0 (Zippin [64]) ("separable" is necessary since c_0 is not complemented in l_∞).

Remark 1.2: We point out that the implication (b) $\Rightarrow$ (c) is strict, otherwise the false equivalence (b) $\Rightarrow$ (d) would follow. Also the implication (e) $\Rightarrow$ (d) (that is (f) $\Rightarrow$ (d)) is strict; otherwise, by Reference IV* of the Introduction, every Banach space would have uniformly complemented subspaces, while for instance, in [35], there is a separable infinite-dimensional Banach space X, with X and X^* of cotype 2 and without the approximation property, where the norm of any rank n projection on it is of order $\sqrt{n}$.

The next proposition is well-known and it will simplify some proofs of §2; it concerns the following definition: If X and Z are Banach spaces we say that Z **is isomorphically finitely represented in** X (X **is isomorphically finite transformable in** X) if there exists a positive number K such that, for any finite-dimensional subspace Z_0 of Z, there exists a subspace X_0 of X and an isomorphism

$$T : X_0 \to Z_0 \ (T : X/X_0 \to Z_0) \quad \text{with } \|T\| \cdot \|T^{-1}\| < K.$$

PROPOSITION 1.2: *For a Banach space X we have the following equivalences:*

(a) *c_0 is isomorphically finitely represented in X*

$\Leftrightarrow$ (b) *c_0 is finitely represented in X.*

(c) *X is isomorphically finitely transformable in c_0*

$\Leftrightarrow$ (d) *X is finitely transformable in c_0.*

(e) *l_1 is isomorphically finitely represented in X*

$\Leftrightarrow$ (f) *l_1 is finitely represented in X.*

Since (a) $\Leftrightarrow$ (b) and (e) $\Leftrightarrow$ (f) are already known ([8], see also [23] and [27]), we only point out that (c) $\Rightarrow$ (d) follows from (c) $\Rightarrow$ (e) (practically the same proof of (c) $\Rightarrow$ (d) of Proposition 1.1 works) and from (e) $\Rightarrow$ (f) and finally from (d) $\Rightarrow$ (c) of Proposition 1.1.

The next proposition gives a characterization of the Banach spaces where l_1 is finitely represented and those which are finitely transformable in l_1. Although we do not need these two characterizations in this paper, we state them for completeness (the proof easily follows from the proof of (d) $\Rightarrow$ (c) of Proposition 1.1).

PROPOSITION 1.3: *Let X be a Banach space, then:*

(a) *l_1 is finitely represented in X*

$\Leftrightarrow$ (b) *Every Banach space is finitely constructible in X.*

(c) *X is finitely transformable in l_1*

$\Leftrightarrow$ (d) *X is finitely transformable in every Banach space.*

Remark 1.3: Proposition 1.3 is a kind of finite version of the well-known fact that every separable Banach space is a quotient of l_1. In particular this proposition corresponds to an analogous property for l_∞:

"l_∞ is finitely represented in $X \Leftrightarrow$ every Banach space is finitely represented in X" (since l_∞ contains copies of any separable Banach space).

Proof of Proposition 1.1: (d) $\Rightarrow$ (c). The first part of our argument is just a finite-dimensional version of the fact that every separable Banach space is a quotient space of l_1. We consider here a slightly more general situation than needed; this will be used later in the proof of Proposition 1.3. Recall that l_1 is finitely represented in the Banach space X and let W be any space. Fix a positive integer m and an m-dimensional subspace Z of W. Using Reference V* of the Introduction, given $\varepsilon > 0$ and letting ε' be a positive number with $(1+\varepsilon')(1+m\varepsilon') < 1+\varepsilon$, we select $\{w_n\}_{n=1}^N$ in Z so that

$$\begin{aligned}&\{w_n\}_{n=1}^m \text{ is an Auerbach basis of } Z \text{ and the sequence}\\ &\{w_n\}_{n=m+1}^N \text{ is a sequence of elements of norm 1}\\ &\text{which is } e'\text{-dense in the unit sphere of } Z.\end{aligned} \tag{1.1}$$

By our assumption, there exists a sequence $\{e_n\}_{n=1}^N$ in X which is $(1+\varepsilon')$-equivalent to the unit vector basis of l_1^N. Let $Y = \{\sum_{n=1}^N b_n e_n : \sum_{n=1}^N b_n w_n = 0\}$. We claim that

$$\operatorname{span}\{e_n\}_{n=1}^N = \operatorname{span}\{e_n\}_{n=1}^m + Y. \tag{1.2}$$

Indeed, it is clear that $(\operatorname{span}\{e_n\}_{n=1}^m) \cap Y = \{0\}$ and, if $m+1 \le n \le N$ and $w_n = -\sum_{k=1}^m a_k w_k$, then $e_n + \sum_{k=1}^m a_k e_k \in Y$ and hence $e_n \in \operatorname{span}\{e_k\}_{k=1}^m + Y$. Our first step is to prove that $\{e_n + Y\}_{n=1}^m$ is $(1+\varepsilon')$-equivalent to $\{w_n\}_{n=1}^m$ (this fact already gives (a) $\Rightarrow$ (b) of Proposition 1.3).

Let us point out that for any sequence $\{b_n\}_{n=1}^N$ of numbers, if $\sum_{n=1}^N b_n w_n = 0$, then $\sum_{n=m+1}^N |b_n| \ge \left\| \sum_{n=m+1}^N b_n w_n \right\| = \left\| \sum_{n=1}^m b_n w_n \right\|$. Therefore, if $\{a_n\}_{n=1}^m$ is any given sequence of numbers, then (in the first part we use the norm of X, in the second part the norm of W):

$$\begin{aligned}\left\| \sum_{n=1}^m a_n(e_n + Y) \right\| &= \inf \left\{ \left\| \sum_{n=1}^m a_n e_n + \sum_{n=1}^N b_n e_n \right\| : \sum_{n=1}^N b_n w_n = 0 \right\}\\ &\ge \frac{1}{1+\varepsilon'} \inf \left\{ \sum_{n=1}^m |a_n + b_n| + \sum_{n=m+1}^N |b_n| : \sum_{n=1}^N b_n w_n = 0 \right\}\\ &\ge \frac{1}{1+\varepsilon'} \inf \left\{ \sum_{n=1}^m |a_n + b_n| + \left\| \sum_{n=1}^m b_n w_n \right\| : \sum_{n=1}^N b_n w_n = 0 \right\}\end{aligned}$$

$$\geq \frac{1}{1+\varepsilon'} \inf\Big\{ \sum_{n=1}^{m} |a_n + b_n| + \Big\| \sum_{n=1}^{m} b_n w_n \Big\| : \{b_n\}_{n=1}^{m} \Big\}$$
$$= \frac{1}{1+\varepsilon'} \Big\| \sum_{n=1}^{m} a_n w_n \Big\|$$

(the last inequality following from the fact that

$$\Big\|\sum_{n=1}^{m} a_n w_n\Big\| \leq \Big\|\sum_{n=1}^{m} (a_n + b_n) w_n\Big\| + \Big\|\sum_{n=1}^{m} b_n w_n\Big\| \leq \sum_{n=1}^{m} |a_n + b_n| + \Big\|\sum_{n=1}^{m} b_n w_n\Big\|).$$

On the other hand, suppose that $a' = \|\sum_{n=1}^{m} a_n w_n\|$ and choose $m < n' \leq N$ such that $\|\sum_{n=1}^{m} a_n w_n - a' w_{n'}\| < \varepsilon' a'$. If $a' w_{n'} = \sum_{n=1}^{m} a'_n w_n$ then, clearly, $a' e_{n'} - \sum_{n=1}^{m} a'_n e_n \in Y$ and, by (1.2) and (1.1),

$$\begin{aligned}
\Big\|\sum_{n=1}^{m} (a_n - a'_n)(e_n + Y)\Big\| &\leq \Big\|\sum_{n=1}^{m} (a_n - a'_n) e_n\Big\| \leq (1+\varepsilon') \sum_{n=1}^{m} |a_n - a'_n| \\
&\leq (1 + e')m \cdot \max\{|a_n - a'_n| : 1 \leq n \leq m\} \\
&\leq (1+\varepsilon')m \Big\|\sum_{n=1}^{m} (a_n - a'_n) w_n\Big\| \\
&= (1+\varepsilon')m \Big\|\sum_{n=1}^{m} a_n w_n - a' w_{n'}\Big\| \leq (1+\varepsilon')m\varepsilon' a'.
\end{aligned}$$

It follows, again, by (1.2) and (1.1), that

$$\begin{aligned}
\Big\|\sum_{n=1}^{m} a_n (e_n + Y)\Big\| &\leq \Big\|\sum_{n=1}^{m} a'_n (e_n + Y)\Big\| + \Big\|\sum_{n=1}^{m} (a_n - a'_n)(e_n + Y)\Big\| \\
&\leq \Big\|\sum_{n=1}^{m} a'_n (e_n + Y)\Big\| + (1+\varepsilon')m\varepsilon' a' \\
&\leq \Big\|\sum_{n=1}^{m} a'_n e_n - \Big\{\sum_{n=1}^{m} a'_n e_n - a' e_{n'}\Big\}\Big\| + (1+\varepsilon')m\varepsilon' a' \\
&\leq (1+\varepsilon')(1+m\varepsilon')a'
\end{aligned}$$

which completes the first step (and hence the proof of (a) $\Rightarrow$ (b) of Proposition 1.3).

By dualizing the fact that $\{e_n + Y\}_{n=1}^{M}$ is $(1+\varepsilon)$-equivalent to the Auerbach basis $\{w_n\}_{n=1}^{m}$ we get that there exists a sequence $\{F_n\}_{n=1}^{m}$ in $(X/Y)^*$ which forms, together with $\{e_n + Y\}_{n=1}^{m}$, a biorthogonal system for which

$$\|F_n\| \leq 1 + \varepsilon \quad \text{for all } 1 \leq n \leq m. \tag{1.3}$$

Now suppose that $W = c_0$ and $\{w_n\}_{n=1}^m$ is the natural basis of l_∞^m. Let T be the quotient mapping of X onto X/Y, let $e_n^* = T^*F_n \in X^*$ for $1 \leq n \leq m$ and put $H = (\operatorname{span}\{e_n^*\}_{n=1}^m)_T$; let S be the quotient mapping of X onto X/H and, for each n with $1 \leq n \leq m$, let $G_n \in (X/H)^*$ such that $e_n^* = S^*G_n$, hence $\|G_n\| = \|F_n\|$; then, since $\{F_n\}_{n=1}^m$ is $(1+\varepsilon)$-equivalent to the unit vectors basis of l_1^m and since $H \supset Y$, we get that for every $\{a_1, \ldots, a_m\} \in l_\infty^m$

$$\begin{aligned}
\max\{|a_n| : 1 \leq n \leq m\} &= \max\{|e_n^*(\sum_{i=1}^m a_i e_i)| : 1 \leq n \leq m\} \\
&= \max\left\{\left|G_n\Big(\sum_{i=1}^m a_i e_i + H\Big)\right| : 1 \leq n \leq m\right\} \\
&\leq (1+\varepsilon)\left\|\sum_{i=1}^m a_i e_i + H\right\| \\
&\leq (1+\varepsilon)\left\|\sum_{i=1}^m a_i e_i + Y\right\| \\
&\leq (1+\varepsilon)^2 \max\{|a_n| : 1 \leq n \leq m\}.
\end{aligned} \tag{1.4}$$

This completes the proof of the implication (d) $\Rightarrow$ (c); therefore we already have (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Leftarrow$ (d) (since the proof of (b) $\Rightarrow$ (c) is analogous to the proof of (b) $\Rightarrow$ (a)).

Regarding (e) $\Rightarrow$ (d) we are going to prove, more in general, that: "l_1 is finitely constructible in $X \Rightarrow l_1$ is finitely represented in X" (hence also (a) $\Leftarrow$ (b) of Proposition 1.3 will be proved).

Therefore assume that in X, for some subspace U,

$$\{e_n + U\}_{n=1}^m \text{ is } (1+\tfrac{1}{2}\varepsilon)\text{-equivalent to the natural basis of } l_1^m;$$

then, for each n with $1 \leq n \leq m$, we can choose e_n in $e_n + U$ such that $\|e_n\| < 1+\varepsilon$; hence for each sequence $\{a_n\}_{n=1}^m$ of numbers it follows that

$$\frac{1}{1+e}\sum_{n=1}^m |a_n| \leq \left\|\sum_{n=1}^m a_n e_n + U\right\| \leq \left\|\sum_{n=1}^m a_n e_n\right\| \leq \sum_{n=1}^m |a_n| \cdot \|e_n\| \leq (1+\varepsilon)\sum_{n=1}^m |a_n|.$$

Then also (e) $\Rightarrow$ (f) (hence (e) $\Leftrightarrow$ (f)) holds because, as indicated above, for each $u \in U$ we have that

$$\left\|\sum_{n=1}^m a_n e_n + u\right\| \geq \left\|\sum_{n=1}^m a_n e_n + U\right\| \geq \frac{1}{1+\varepsilon}\sum_{n=1}^m |a_n| \geq \frac{1}{(1+\varepsilon)^2}\left\|\sum_{n=1}^m a_n e_n\right\|.$$

Finally, regarding (c) $\Rightarrow$ (d), we only recall (see for instance [26], proof of corollary 1.f.13, pp. 92–93) that, in the real case, if for a subspace U of $X\{e'_n + U\}_{n=1}^{2^m}$ is $(1+\varepsilon)$-equivalent to the natural basis of $l_\infty^{2^m}$, there exists $\{e_n\}_{n=1}^m$ in $\mathrm{span}\{e'_n\}_{n=1}^{2^m}$ such that $\{e_n + U\}_{n=1}^m$ is $(1+\varepsilon)$-equivalent to the natural basis of l_1^m (precisely e_n are the vectors

$$\begin{aligned} e_1 &= e'_1 + \cdots + e'_{2^{m-1}} - e'_{2^{m-1}+1} - \cdots - e'_{2^m}, \\ &\cdots \\ e_m &= e'_1 - e'_2 + e'_3 - e'_4 + \cdots + e'_{2^m-1} - e'_{2^m}); \end{aligned} \tag{1.5}$$

now it is sufficient to use the above proof of (e) $\Rightarrow$ (d); in the complex case the procedure is analogous.

This completes the proof of Proposition 1.1.

Proof of Proposition 1.3: We have already proved (a) $\Rightarrow$ (b) during the proof of (d) $\Rightarrow$ (c) of Proposition 1.1 and (a) $\Leftarrow$ (b) during the proof of (e) $\Rightarrow$ (d) of Proposition 1.1; therefore we have only to prove (c) $\Rightarrow$ (d).

Assume X finitely transformable in l_1, consider any Banach space W, fix $\varepsilon > 0$ and any finite m-dimensional subspace Z of W. Then we have again (1.1) and, by our assumption, there exist a subspace V and a sequence $\{e_n\}_{n=1}^N$ in X such that $\{e_n + V\}_{n=1}^N$ is $(1+\varepsilon')$-equivalent to the unit vectors basis of l_1^N, with $X = \mathrm{span}\{e_n\}_{n=1}^N + V$; therefore (1.2) becomes

$$X/V = \mathrm{span}\{e_n + V\}_{n=1}^N = Y/V + \mathrm{span}\{e_n + V\}_{n=1}^m$$

with $Y = \{\sum_{n=1}^N b_n e_n : \sum_{n=1}^N b_n w_n = 0\}$; that is, setting $U = Y + V$, $X = \mathrm{span}\{e_n\}_{n=1}^m + U$. Moreover, by the first part of the proof of (d) $\Rightarrow$ (c) of Proposition 1.1, we have that

$$\begin{aligned} \frac{1}{1+\varepsilon}\Big\|\sum_{n=1}^m a_n w_n\Big\| &\le \Big\|\sum_{n=1}^m a_n(e_n + V) + Y/V\Big\| \Big(= \Big\|\sum_{n=1}^m a_n e_n + U\Big\|\Big) \\ &\le (1+\varepsilon)\Big\|\sum_{n=1}^m a_n w_n\Big\| \end{aligned}$$

for any $\{a_n\}_{n=1}^m$ of numbers; that is X is finitely transformable in X, which completes the proof of Proposition 1.3.

Finally we point out the following fact:

LEMMA 1.1: *For each positive integer m there exists another integer $M(m)$ such that, if $\{e_k\}_{k=1}^{2^{M(m)}}$ is the natural basis of $l_\infty^{2^{M(m)}}$, then $l_\infty^{2^{M(m)}}$ has an Auerbach basis $\{x_n\}_{n=1}^{2^{M(m)}} = W_{M(m)}(\{e_k\}_{k=1}^{2^{M(m)}})$ which is $2^{M(m)-3m-3}$-equivalent to the natural basis of $l_1^{2^{M(m)}}$.*

The proof follows from a more general fact of [15], where $M(m)$ is a suitable integer such that $\{x_n\}_{n=1}^{2^{M(m)}}$, defined by means of the "Walsh-elements" on $\{e_k\}_{k=1}^{2^{M(m)}}$ (analogously to (1.5) where the e_n were the "Rademacher elements" on $\{e'_n\}_{n=1}^{2^m}$), is $2^{M(m)-3m-3}$-equivalent to the natural basis of $l_1^{2^{M(m)}}$. (Lemma 1.1 has an easy, direct proof which is also valid in the complex case.) In this paper we wish only to repeat the definition of $\{x_n\}_{n=1}^{2^{M(m)}}$: For each p, k with $1 \le p \le M(m)$ and $1 \le k \le 2^p$ we set

$$B_{p,k} = \sum_{s=(k-1)2^{M(m)-p}+1}^{k\cdot 2^{M(m)-p}} \qquad (\text{ then, clearly, } B_{p,k} = B_{p+1,2k-1} + B_{p+1,2k}).$$

We start with $x_1 = B_{1,1} + B_{1,2}$ and $x_2 = B_{1,1} - B_{1,2}$ and we write $\{x_n\}_{n=1}^{2^1} = W_1(\{B_{1,k}\}_{k=1}^{2^1})$. Now we proceed by induction: Suppose that $1 \le p \le M(m) - 1$ and $\{x_n\}_{n=1}^{2^p} = W_p(\{B_{p,k}\}_{k=1}^{2^p})$ where, for each n with $1 \le n \le 2^p$,

$$x_n = \sum_{k=1}^{2^p} \theta_{p,n,k} B_{p,k} = \sum_{k=1}^{2^p} \theta_{p,n,k}(B_{p+1,2k-1} + B_{p+1,2k})$$

with $\theta_{p,n,k} \in \{-1, 1\}$ for $1 \le k \le 2^p$. Then we set

$$\{x_n\}_{n=2^p+1}^{2^{p+1}} = W_p(\{B_{p+1,2k-1} - B_{p+1,2k}\}_{k=1}^{2^p}),$$

that is we set, again for each n with $1 \le n \le 2^p$,

$$x_{2^p+n} = \sum_{k=1}^{2^p} \theta_{p,n,k}(B_{p+1,2k-1} - B_{p+1,2k}).$$

Hence we have defined $\{x_n\}_{n=1}^{2^{p+1}} = W_{p+1}(\{B_{p+1,k}\}_{k=1}^{2^{p+1}})$ and, if $p = M(m) - 1$, $\{x_n\}_{n=1}^{2^{M(m)}} = W_{M(m)}(\{e_k\}_{k=1}^{2^{M(m)}})$. From this definition it easily follows that $x_n = \sum_{k=1}^{2^{M(m)}} \theta_{n,k} e_k$, with $\theta_{n,k} \in \{-1, 1\}$ for $1 \le k \le 2^{M(m)}$, for each n with $1 \le n \le 2^{M(m)}$; moreover, for each n', n'' with $1 \le n' \ne n'' \le 2^{M(m)}$,

$$\{k\}_{k=1}^{2^{M(m)}} = \{k'\}_{k=1}^{2^{M(m)-1}} \cup \{k''\}_{k=1}^{2^{M(m)-1}}$$

with $\theta_{n',k'} = \theta_{n'',k'}$ and $\theta_{n',k''} = -\theta_{n'',k''}$ for $1 \le k \le 2^{M(m)-1}$; from this fact it is easy to obtain that $\{x_n\}_{n=1}^{2^{M(m)}}$ is an Auerbach basis of $l_\infty^{2^{M(m)}}$. This completes the proof of Lemma 1.1.

Remark 1.4: In the next paragraph we shall have to construct, in a Banach space X where l_1 is finitely represented, a sequence $\{V_m\}$ of subspaces of X with the properties of (4) of the Introduction. We recall that, if X_m is a finite m-dimensional subspace of X, then $\lambda(X_m) < \sqrt{m}$ (for a better estimate see [21], an improvement of previous estimates again of Konig–Tomczak Jaegermann (1990), Lewis (1988), Konig–Lewis (1987) and Kadec–Snobar (1971)). From this fact it is easy to obtain (a) of (4) of the Introduction, with the further condition of $H_m < (r(m))^{1/2}$; but the essential difficulty lies in the proof of (b) of (4). Therefore, as we have already said in the Introduction, we shall prefer to start from (d) $\Rightarrow$ (c) of Proposition 1.1 and afterward we shall use Lemma 1.1.

2. Existence of the M-basis with controlled coefficients

The next lemma is obvious and we give its proof only for completeness.

LEMMA 2.1: *Let X be a Banach space with the following property: there exist $p \geq 1$ and two positive numbers H and K such that, for any positive integer r, there exist a subspace X_r of X with $d(X_r, l_p^r) < H$ and a projection P_r of X onto X_r with $\|P_r\| < K$. Then, if U is a q-codimensional subspace of X with q finite, for any positive integer m, there is a subspace U_m of U with $d(U_m, l_p^m) < H$ and a projection $P'_m : U \to U_m$ with $\|P'_m\| < HK$.*

Proof: By hypothesis, for any positive integer r, there exists a sequence $\{v_{r,n}\}_{n=1}^r$ so that

$$(2.1)\quad \begin{aligned} &X \supset X_r = \operatorname{span}\{v_{r,n}\}_{n=1}^r \text{ and there is an isomorphism}\\ &T_r : X_r \to l_p^r \text{ with } \|T_r\| \cdot \|T_r^{-1}\| < H;\\ &T_r(v_{rn}) = e_n \text{ for } 1 \leq n \leq r \text{ where } \{e_n\}_{n=1}^r \text{ is the natural basis of } l_p^r;\\ &\text{moreover, there is a projection } P_r : X \to X_r \text{ with } \|P_r\| < K. \end{aligned}$$

Let U be a q-codimensional subspace of X with q finite Fix a positive integer m and set

$$(2.2)\quad \begin{gathered} g(0,q) = 0 \text{ and, for each } n \geq 1, g(n,q) = g(n-1,q) + q + n,\\ \text{hence } g(n,q) = nq + \sum_{k=1}^{n} k. \end{gathered}$$

By (2.1) we can choose a subspace V_0 of $U \cap X_{g(m,q)}$ with $\dim V_0$(= dimension of V_0) $= g(m,q) - q$. By (2.2), $g(m,q) - \{g(m,q) - q - 1\} = g(1,q)$, hence by (2.1) we can set $V_0 = \operatorname{span}\{u_1\} + V_{0,1}$ with $u_1 \in \operatorname{span}\{v_{g(m,q),n}\}_{n=1}^{g(1,q)}$ and with $\dim V_{0,1} =$

$g(m,q)-q-1$ (indeed if $V_0 = \text{span}\{w_k\}_{k=1}^{g(m,q)-q}$ where $w_k = \sum_{n=1}^{g(m,q)} a_{kn} v_{g(m,q),n}$ for each k with $1 \le k \le g(m,q)-q$, if there exists $k(1)$, with $1 \le k(1) \le g(m,q)-q$ and $a_{k(1),g(m,q)} \neq 0$, we set, for each k with $1 \le k(\neq k(1)) \le g(m,q)-q, w'_k = w_k - (a_{k,g(m,q)}/a_{k(1),g(m,q)})w_{k(1)}$, hence $\{w_k^{(1)}\}_{k=1}^{g(m,q)-q-1} = \{w'_k\}_{k(\neq k(1))=1}^{g(m,q)-q}$ is a sequence of $\text{span}\{v_{g(m,q),n}\}_{n=1}^{g(m,q)-1}$; the same procedure gives $\{w_k^{(2)}\}_{k=1}^{g(m,q)-q-2}$ in $\text{span}\{v_{g(m,q),n}\}_{n=1}^{g(m,q)-2}$ and so on, altogether for $g(m,q)-q-1$ times). By the same procedure, since by (2.2)

$$\begin{aligned} g(m,q) - \{\dim V_{01} - g(1,q) - 1\} &= g(m,q) - \{[g(m,q) - q - 1] - q(1,q) - 1\} \\ &= g(1,q) + q + 2 = g(2,q), \end{aligned}$$

by (2.1) we can set $V_{0,1} = \text{span}\{u_2\} + V_{0,2}$ with $u_2 \in \text{span}\{v_{g(m,q),n}\}_{n=g(1,q)+1}^{g(2,q)}$ and with $\dim V_{0,2} = g(m,q) - q - 2$, and so on.

Proceeding in this way, altogether for m times, we obtain the subspace $U_m = \text{span}\{u_n\}_{n=1}^m$ of V_0, where $u_n \in \text{span}\{v_{g(m,q),k}\}_{k=g(n-1,q)+1}^{g(n,q)}$ for $1 \le n \le m$ (that is $\{u_n\}_{n=1}^m$ is a **block sequence** of $\{v_{g(m,q),n}\}_{n=1}^{g(m,q)}$). Therefore by (2.1), $\{T_{g(m,q)}(u_n)\}_{n=1}^m$ is a block sequence of $\{e_n\}_{n=1}^{g(m,q)}$, hence ([25] p. 53) it follows that $\text{span}\{T_{g(m,q)}(u_n)\}_{n=1}^m$ is isometric to l_p^m and there is a projection $P''_m : l_p^{g(m,q)} \to \text{span}\{T_{g(m,q)}(u_n)\}_{n=1}^m$ with $\|P''_m\| = 1$. Setting $T'_m = T_{g(m,q)|_{U_m}}, P'''_m = T^{-1}_{g(m,q)} \cdot P''_m \cdot T_{g(m,q)} \cdot P_{g(m,q)}$ and $P'_m = P'''_{m|_U}$, we have that T'_m is an isomorphism $U_m \to l_p^m$ with $\|T'_m\| \cdot \|T'^{-1}_m\| < H, P'''_m$ is a projection $X \to U_m$ with $\|P'''_m\| < HK, P'_m$ is a projection $U \to U_m$ with $\|P'_m\| \le \|P'''_m\|$. This completes the proof of Lemma 2.1.

We start with the Banach spaces where l_1 is finitely represented.

The next lemma improves (d) $\Rightarrow$ (c) of Proposition 1.1.

LEMMA 2.2: *If l_1 is finitely represented in a Banach space X, then, for any finite-dimensional subspace Y of $X, X/Y$ is finitely transformable in c_0.*

Proof: Let Y be a finite-dimensional subspace of X; by (e) $\Rightarrow$ (f) of Proposition 1.2 and (d) $\Rightarrow$ (c) of Proposition 1.1 it is sufficient to see that l_1 is isomorphically finitely represented in X/Y. Then let U be a subspace of X such that $Y+U = X$ with $Y \cap U = \{0\}$: since U has finite codimension, by the proof of Lemma 2.1, l_1 is finitely represented in U; on the other hand, since Y has finite dimension, X/Y and U are isomorphic; hence l_1 is isomorphically finitely represented in X/Y, which completes the proof of Lemma 2.2.

LEMMA 2.3: *Let X be a Banach space where l_1 is finitely represented and let $\{y_n, y_n^*\}_{n=1}^Q$ be a biorthogonal system of X. Fix $\varepsilon > 0$ and a positive integer m.*

Then there exist $\{x_n\}_{n=1}^T$ *in* X *and* $\{x_n^*\}_{n=1}^T$ *in* X^* *such that (see Lemma 1.1):*

$$\{y_n, y_n^*\}_{n=1}^Q \cup \{x_n, x_n^*\}_{n=1}^T \text{ is biorthogonal with } \|x_n\| = 1 \text{ and } \|x_n^*\| < 2+\varepsilon \text{ for } 1 \le n \le T, \text{ where } T = 2^{M(m)}; \text{ moreover, setting } X \cap \Big\{ \bigcap_{n=1}^{T} x_{n\perp}^* \Big\} = U(\text{ hence } X = \operatorname{span}\{x_n\}_{n=1}^T + U), \{x_n + U\}_{n=1}^T \text{ is } 2^{M(m)-3m-1}\text{-equivalent to the natural basis of } l_1^{2^{M(m)}}.$$

Proof: Using Lemma 2.2 and letting ε' be a positive number such that $2(1+3\varepsilon')^2 < 2+\varepsilon$ and $(1+3\varepsilon')^2 < 2$, we can select, for any positive integer R, $\{u_n'\}_{n=1}^R$ in X and $\{u_n'^*\}_{n=1}^R$ in X^* such that

$$\begin{aligned} &\{u_n', u_n'^*\}_{n=1}^R \text{ is biorthogonal and, setting } U' = X \cap \Big\{ \bigcap_{n=1}^{R} u_{n\perp}'^* \Big\} \\ &\text{and } W' = U' \cap \Big\{ \bigcap_{n=1}^{Q} y_{n\perp}^* \Big\},\ U' \supset Y = \operatorname{span}\{y_n\}_{n=1}^Q, \\ &\text{hence } U' = Y + W' \text{ and } \{u_n' + U'\}_{n=1}^R \text{ is} \\ &(1+\varepsilon')\text{-equivalent to the natural basis of } l_\infty^R. \end{aligned} \tag{2.3}$$

(Indeed by Lemma 2.2 there exists in X/Y a biorthogonal system $\{u_n'+Y, F_n\}_{n=1}^R$ such that, setting $U'' = (X/Y) \cap \{\bigcap_{n=1}^R F_{n\perp}\}, \{(u_n' + Y) + U''\}_{n=1}^R$ is $(1+\varepsilon')$-equivalent to the natural basis of l_∞^R; setting now $u_n'^*(x) = F_n(x+Y)$ for each x of X and for $1 \le n \le R$, moreover $U' = X \cap \{\bigcap_{n=1}^R u_{n\perp}'^*\}$, it easily follows that $U' \supset Y$ and $U'' = U'/Y$; on the other hand $(X/Y)/U'' = (X/Y)/(U'/Y)$ and X/U' are isometric, therefore also $\{u_n' + U'\}_{n=1}^R$ is $(1+\varepsilon')$-equivalent to the natural basis of l_∞^R.)

We shall use the following two easy facts:

$$\begin{aligned} &\text{If } \{v_n\}_{n=1}^N \text{ is } (1+\varepsilon')\text{-equivalent to the natural basis of } l_\infty^N, \text{ setting} \\ &\{n\}_{n=1}^N = \{n_k'\}_{k=1}^{N'} \cup \{n_k''\}_{k=1}^{N''}, \text{ we have that } \{v_{n_k'} + \operatorname{span}\{v_{n_i''}\}_{i=1}^{N''}\}_{k=1}^{N'} \\ &\text{is still } (1+\varepsilon')\text{-equivalent to the natural basis of } l_\infty^{N'} \end{aligned} \tag{2.4}$$

(indeed, for any $\{a_{n_k'}\}_{k=1}^{N'}$ of numbers with $\max\{|a_{n_k'}l : 1 \le k \le N'\} = 1$,

$$\frac{1}{1+\varepsilon'} \le \Big\| \sum_{k=1}^{N'} a_{n_k'} v_{n_k'} + \operatorname{span}\{v_{n_i''}\}_{i=1}^{N''} \Big\| \le \Big\| \sum_{k=1}^{N'} a_{n_k'} v_{n_k'} \Big\| \le 1+\varepsilon').$$

Moreover, let $\{v_n\}_{n=1}^N$ be a sequence of X and let $\{w_n\}_{n=1}^N$ be any sequence of any normed space, let V, V' and V'' be subspaces of X such that:

$$\begin{gathered} V' \supset V \supset V'', \text{ moreover } \{v_n + V'\}_{n=1}^N \text{ and } \{v_n + V''\}_{n=1}^N \text{ are} \\ H\text{-equivalent to } \{w_n\}_{n=1}^N; \\ \text{then } \{v_n + V\}_{n=1}^N \text{ too is } H\text{-equivalent to } \{w_n\}_{n=1}^N \end{gathered} \tag{2.5}$$

(indeed, for any $\{a_n\}_{n=1}^N$ of numbers,

$$\begin{aligned} \frac{1}{H}\Big\|\sum_{n=1}^N a_n w_n\Big\| &\le \Big\|\sum_{n=1}^N a_n v_n + V'\Big\| \le \Big\|\sum_{n=1}^N a_n v_n + V\Big\| \\ &\le \Big\|\sum_{n=1}^N a_n v_n + V''\Big\| \le H\Big\|\sum_{n=1}^N a_n w_n\Big\|). \end{aligned}$$

Proceeding in this way, since Y has finite dimension, for each fixed positive integer T, there exists another positive integer $R(T)$ such that, if in (2.3) $R = R(T)$, then there exists a subsequence of $2T$ elements of $\{n\}_{n=1}^R$, which we may call $\{n\}_{n=1}^{2T}$ without loss of generality, such that the following two properties hold:

(i) Setting $u_n' = y_n' + u_n''$ with $y_n' \in Y$ and $u_n'' \in X \cap \{\bigcap_{n=1}^Q y_{n\perp}^*\}$ for $1 \le n \le R$, we have that $\|y_n' - y_{n+T}'\| < \varepsilon'/T$ for $1 \le n \le T$.

(ii) For any fixed $\{a_n\}_{n=1}^T$ of numbers with $\max\{|a_n| : 1 \le n \le T\} = 1$, there exist y and y' in Y, with $\|y - y'\| < \varepsilon'$, such that

$$\Big\|\sum_{n=1}^T a_n u_n' + U'\Big\| = \Big\|\sum_{n=1}^T a_n u_n' + Y + W'\Big\| = \Big\|\sum_{n=1}^T a_n u_n' + y + W'\Big\|,$$

$$\begin{aligned} \Big\|\sum_{n=T+1}^{2T} a_{n-T} u_n' + U'\Big\| &= \Big\|\sum_{n=T+1}^{2T} a_{n-T} u_n' + Y + W'\Big\| \\ &= \Big\|\sum_{n=T+1}^{2T} a_{n-T} u_n' + y' + W'\Big\| \end{aligned}$$

(by (2.3) we may assume $\|u_n'\| < 2$ for $1 \le n \le R$, hence, setting

$$\inf\{\operatorname{dist}(y, X \cap \{\bigcap_{n=1}^Q y_{n\perp}^*\}) : y \in Y, \|y\| = 1\} = d > 0,$$

the following holds: In (i), $\|y_n'\| < 2/d$ for $1 \le n \le R$; in (ii), by our assumptions the set $\{\sum_{k=1}^T a_k u_{n_k}' : \max\{|a_k| : 1 \le k \le T\} = 1, \{n\}_{n=1}^R \supset \{n_k\}_{k=1}^T\}$ is norm bounded by $2T$, hence the set of all the possible y, y' which appear in (ii) is norm bounded by $2T/d$, since $X \cap \{\bigcap_{n=1}^{Q} y^*_{n\perp}\} \supset W'$; now it is sufficient to point out that T and d do not depend on R). We are going to prove that, setting

$$u_n = u''_n - u''_{n+T} \text{ and } u^*_n = u'^*_n \text{ for } 1 \le n \le T,\ U = X \cap \Big\{ \bigcap_{n=1}^{T} u^*_{n\perp} \Big\}$$

(2.6) and $W = U \cap \Big\{ \bigcap_{n=1}^{Q} y^*_{n\perp} \Big\}$, if follows that $\{y_n, y^*_n\}_{n=1}^{Q} \cup \{u_n, u^*_n\}_{n=1}^{T}$ is biorthogonal, hence $U = Y + W$, moreover, $\{u_n + U\}_{n=1}^{T}$ and $\{u_n + W\}_{n=1}^{T}$ are, respectively , $(1+\varepsilon')$-equivalent and $2(1+\varepsilon')$-equivalent to the natural basis of l^T_∞.

By (i) and by the definition of $\{u_n\}_{n=1}^{T}, X \cap \{\bigcap_{n=1}^{Q} y^*_{n\perp}\} \supset \{u_n\}_{n=1}^{T}$; moreover, by (2.3), by (i) and by the definition of $\{u_n, u^*_n\}_{n=1}^{T}$,

$$u^*_m(u_n) = u'^*_m(u''_n - u''_{n+T}) = u'^*_m((u'_n - u'_{n+T}) - (y'_n - y'_{n+T})) = u'^*_m(u'_n) \quad \text{for } 1 \le m, n \le T;$$

hence by (2.3) it follows that $\{y_n, y^*_n\}_{n=1}^{Q} \cup \{u_n, u^*_n\}_{n=1}^{T}$ is biorthogonal. Let us point out that

$$U = U' + \operatorname{span}\{u'_n\}_{n=T+1}^{R} \quad \text{and} \quad U \supset W \supset W' \tag{2.7}$$

(indeed, for the first one, since by (2.3) $\{u'_n, u'^*_n\}_{n=1}^{R}$ is biorthogonal, by the definitions of (2.6), $U = X \cap \{\bigcap_{n=1}^{T} u'^*_{n\perp}\} = X \cap \{\bigcap_{n=1}^{R} u'^*_{n\perp}\} + \operatorname{span}\{u'_n\}_{n=T+1}^{R}$, so we may use now the definition of U' in (2.3); while for the second one, by the definitions of W in (2.6) and of W' in (2.3), $U \supset W = U \cap \{\bigcap_{n=1}^{Q} y^*_{n\perp}\} = \{U' + \operatorname{span}\{u'_n\}_{n=T+1}^{R}\} \cap \{\bigcap_{n=1}^{Q} y^*_{n\perp}\} = W' + \{\operatorname{span}\{u'_n\}_{n=T+1}^{R}\} \cap \{\bigcap_{n=1}^{Q} y^*_{n\perp}\} \supset W')$ (we assume that the notation $A \supset B$ includes the case $A = B$).

By (2.7) and (2.4), $\{u'_n + U\}_{n=1}^{T}$ is $(1+\varepsilon')$-equivalent to the natural basis of l^T_∞; on the other hand, since $U \supset U' \supset Y$, by (i)

$$u'_n + U = (u'_n - u'_{n+T}) - (y'_n - y'_{n+T}) + U = u_n + U \quad \text{for } 1 \le n \le T;$$

hence $\{u_n + U\}_{n=1}^{T}$ too is $(1+\varepsilon')$-equivalent to the natural basis of l^T_∞. Therefore, in order to prove that $\{u_n + W\}_{n=1}^{T}$ is $2(1+2\varepsilon')$-equivalent to the natural basis of l^T_∞, by (2.5) and by the second relation of (2.7), it is sufficient to check that $\{u_n + W'\}_{n=1}^{T}$ is $2(1+2\varepsilon')$-equivalent to the natural basis of l^T_∞. Indeed, for any fixed $\{a_n\}_{n=1}^T$ of numbers with $\max\{|a_n| : 1 \leq n \leq T\} = 1$, using the elements y and y' of (ii), by (2.3) and by (i) we have that

$$
\begin{aligned}
\frac{1}{1+\varepsilon'} \leq & \Big\| \sum_{n=1}^{T} a_n u'_n - \sum_{n=T+1}^{2T} a_{n-T} u'_n + U' \Big\| = \Big\| \sum_{n=1}^{T} a_n (u'_n - u'_{n+T}) + U' \Big\| \\
= & \Big\| \sum_{n=1}^{T} a_n u_n + U' \Big\| \leq \Big\| \sum_{n=1}^{T} a_n u_n + W' \Big\| \\
= & \Big\| \Big\{ \sum_{n=1}^{T} a_n u'_n + W' \Big\} - \Big\{ \sum_{n=1}^{T} a_n u'_{n+T} + W' \Big\} \\
& \qquad - \Big\{ \sum_{n=1}^{T} a_n (y'_n - y'_{n+T}) + W' \Big\} \Big\| \\
= & \Big\| \Big\{ \sum_{n=1}^{T} a_n u'_n + y + W' \Big\} - \Big\{ \sum_{n=1}^{T} a_n u'_{n+T} + y' + W' \Big\} - \{y - y'\} \\
& \qquad - \Big\{ \sum_{n=1}^{T} a_n (y'_n - y'_{n+T}) + W' \Big\} \Big\| \\
\leq & \Big\| \sum_{n=1}^{T} a_n u'_n + y + W' \Big\| \\
& + \Big\| \sum_{n=1}^{T} a_n u'_{n+T} + y' + W' \Big\| + \|y - y'\| \\
& + \Big\| \sum_{n=1}^{T} a_n (y'_n - y'_{n+T}) \Big\| \\
= & \Big\| \sum_{n=1}^{T} a_n u'_n + U' \Big\| + \Big\| \sum_{n=1}^{T} a_n u'_{n+T} + U' \Big\| \\
& + \|y - y'\| + \Big\| \sum_{n=1}^{T} a_n (y'_n - y'_{n+T}) \Big\| \\
\leq & (1+\varepsilon') + (1+\varepsilon') + (\varepsilon') + \sum_{n=1}^{T} |a_n| \cdot \|y'_n - y'_{n+T}\| \\
\leq & 2(1+\varepsilon') + \varepsilon' + \frac{\varepsilon'}{T} \sum_{n=1}^{T} |a_n| \leq 2(1+\varepsilon') + \varepsilon' + \varepsilon' \max\{|a_n| : 1 \leq n \leq T\} \\
= & 2(1+\varepsilon') = 2\varepsilon';
\end{aligned}
$$

that is

$$\frac{1}{1+\varepsilon'} \leq \Big\| \sum_{n=1}^{T} a_n u_n + W' \Big\| \leq 2(+2\varepsilon').$$

This completes the proof of (2.6).

At this point, according to Lemma 1.1 for $T = 2^{M(m)}$ and setting $\{x'_n\}_{n=1}^T = W_{M(m)}(\{u_k\}_{k=1}^T)$, moreover keeping in mind that $\{x_n\}_{n=1}^{2^{M(m)}}$ of Lemma 1.1 is an Auerbach basis, by (2.6) we have that there exist $\{G_n\}_{n=1}^T$ in $(X/U)^*$ and $\{x'^*_n\}_{n=1}^T$ in X^* such that: $\operatorname{span}\{u_n\}_{n=1}^T \supset \{x'_n\}_{n=1}^T; \{x'_n + U\}_{n=1}^T$ and $\{x'_n + W\}_{n=1}^T$ are, respectively, $2^{M(m)-3m-3}(1+\varepsilon')$-equivalent and $2^{M(m)-3m-3}2(1+2\varepsilon')$-equivalent to the natural basis of $l_1^T; \{x'_n, x'^*_n\}_{n=1}^T$ and $\{x'_n + U, G_n\}_{n=1}^T$ are biorthogonal; moreover, for each n with $1 \leq n \leq T$:

$$\frac{1}{1+\varepsilon'} \leq \|x'_n + U\| \leq 1+\varepsilon', \frac{1}{2(1+2\varepsilon')} \leq \|x'_n + W\| \leq 2(1+2\varepsilon'); x'^*_n(x) = G_n(x+U)$$

for each x of $X, \|x'^*_n\| = \|G_n\| \leq 1+\varepsilon'$ and $x'^*_{n\perp} \supset U \supset Y$.

By what we have specified above, $\{y_n, y^*_n\}_{n=1}^Q \cup \{x'_n, x'^*_n\}_{n=1}^T$ is biorthogonal. Now the reason for the presence of $\{u_n + W\}_{n=1}^T$ in (2.6), and hence that of $\{x'_n + W\}_{n=1}^T$, becomes clear: For each n with $1 \leq n \leq T$ there exists w_n in W such that, setting $x''_n = x'_n + w_n$ (hence $x''_n + U = x'_n + U$), we get that

$$\frac{1}{1+\varepsilon'} \leq \|x'_n + U\| = \|x''_n + U\| \leq \|x''_n\| < 2(1+3\varepsilon').$$

Therefore, since $X \cap \{\bigcap_{n=1}^Q y^*_{n\perp}\} \supset W, \{y_n, y^*_n\}_{n=1}^Q \cup \{x''_n, x'^*_n\}_{n=1}^T$ remains biorthogonal and $\{x''_n + U\}_{n=1}^T$ remains $2^{M(m)-3m-3}(1+\varepsilon')$-equivalent to the natural basis of l_1^T. It is now clear that the introduction of $\{u_n + W\}_{n=1}^T$ was necessary in order to get $\{\|x''_n\|\}_{n=1}^T$ to be bounded by $2(1+3\varepsilon')$. Finally, setting for each n with $1 \leq n \leq T, x_n = x''_n/\|x''_n\|$ and $x^*_n = x'^*_n\|x''_n\|$, by our assumption on ε', the assertion is verified.

This completes the proof of Lemma 2.3.

Passing to the Banach spaces of type > 1, we shall prove an analogous property.

LEMMA 2.4: *Let X be a Banach space of type > 1. There exists a positive integer H such that, for each positive integer m and for each biorthogonal system $\{y_n, y^*_n\}_{n=1}^Q$ of X, we can select $\{x_n\}_{n=1}^T$ in X and $\{x^*_n\}_{n=1}^T$ in X^* such that $\{y_n, y^*_n\}_{n=1}^Q \cup \{x_n, x^*_n\}_{n=1}^T$ is biorthogonal with $\|x_n\| = 1$ and $\|x^*_n\| < H$ for $1 \leq n \leq T$; moreover, setting $U = X \cap \{\bigcap_{n=1}^T x^*_{n\perp}\}$ (hence $X = \operatorname{span}\{x_n\}_{n=1}^T +$*

U), $\{x_n\}_{n=1}^{T}$ is $(1+1/2^m)$-equivalent to the natural basis of l_2^T and there exists a positive number $d=d(T)$, such that

$$T \geq 2^m\Big\{\frac{2^{2m+1}}{d}\Big\}^2$$

and for each sequence $\{a_n\}_{n=1}^{T}$ of numbers,

$$\operatorname{dist}\Big(\sum_{n=1}^{T} a_n x_n, U\Big) \geq d\Big\|\sum_{n=1}^{T} a_n x_n\Big\|.$$

Proof: By Reference IV* of the Introduction (see also Lemma 2.1) there exists an integer H (which does not depend on $\{y_n\}_{n=1}^{Q}$) such that

(2.8) setting $Y=\operatorname{span}\{y_n\}_{n=1}^{Q}$ and $V = X\cap\Big\{\bigcap_{n=1}^{Q} y_{n\perp}^*\Big\}$ (thus getting $X = Y+V$), for each positive integer k we have that: $V + V_k + V_{0,k}$ with $V_k=\operatorname{span}\{v_{k,n}\}_{n=1}^{k}$, where $\{v_{k,n}\}_{n=1}^{k}$ is $(1+1/2^k)$-equivalent to the natural basis of l_2^k; moreover, there exists a projection $Q_k\colon V\to V_k$ with $\|Q_k\|<H/8$ and $V_{0,k}=Q_{k\perp}$; finally, $\inf\{\operatorname{dist}(x, Y+V_{0,k})\colon x\in V_k \text{ with } \|x\|=1\} = 4d>0$

(that is, d does not depend on k: Indeed, since $Y\cap V=\{0\}$ and Y has finite dimension, there exists a positive number d' such that, for each v of V and for each y of Y, $\operatorname{dist}(v,Y)\geq d'\|v\|$ and $\operatorname{dist}(y,V)\geq d'\|y\|$, hence for each x of V_k we have that $\operatorname{dist}(x,Y+V_{0,k})\geq d'\cdot\operatorname{dist}(x,V_{0,k})\geq d'(8/H)\|x\|$). We fix a suitable positive integer T and, for each positive integer R (which will be better defined later), two further positive integers T' and T'', as follows:

$$\text{(2.9)}\qquad T\geq 2^m\Big\{\frac{2^{2m+1}}{d}\Big\}^2,\quad T'=RT\quad\text{and}\quad T''=\sum_{i=1}^{T'}[Q+2(i-1)+1].$$

Using (2.8) for $k=T''$, (2.9) and Reference III* of the Introduction, we can select $x_1'\in\operatorname{span}\{v_{T'',n}\}_{n=1}^{Q+1}$ and $x_1'^*\in X^*$, with $\|x_1'\|=\|x_1'^*\|=1=x_1'^*(x_1')$, such that $x_{1\perp}'^*\supset Y$; by the same procedure we can select $x_2'\in\operatorname{span}\{v_{T'',n}\}_{n=Q+2}^{2Q+4}$ and $x_2'^*\in X^*$, with $\|x_2'\|+\|x_2'^*\|=1=x_2'^*(x_2')$ such that $x_2'\in x_{1\perp}'^*$ and $x_{2\perp}'^*\supset Y+\operatorname{span}\{x_1'\}$; hence, proceeding in this way, by (2.9) we are able to define

(2.10) $\{x_n'\}_{n=1}^{T'}$ block sequence of $\{v_{T'',n}\}_{n=1}^{T''}$ such that, for each n with $1\leq n\leq T'$, $\operatorname{dist}(x_n', Y+\operatorname{span}\{x_k'\}_{k(\neq n)=1}^{T'})=\|x_n'\|=1$.

Moreover, there exists a projection

$$Q' : V_{T''} (= \mathrm{span}\{v_{T'',n}\}_{n=1}^{T''}) \to \mathrm{span}\{x'_n\}_{n=1}^{T'} \quad \text{with } \|Q'\| < 2 \tag{2.11}$$

(by (2.8) there is an isomorphism $F : V_{T''} \to l_2^{T''}$ with $\|F\| \cdot \|F^{-1}\| < (1 + 1/2^{T''})^2$). We state that, if R (hence T') is sufficiently large, there exists a partition (see (2.10) and (2.11))

$$\begin{gathered} \{x'_n\}_{n=1}^{T'} = \{x_n\}_{n=1}^{T} \cup \{x''_n\}_{n=1}^{T'''} \text{ such that, setting } X' = \mathrm{span}\{x_n\}_{n=1}^{T}, \\ W = V_{0,T''} + Q'_{\perp} \cap V_{T''} + \mathrm{span}\{x''_n\}_{n=1}^{T'''} \text{ and } U + Y + W, \text{ there exists} \\ \{x^*_n\}_{n=1}^{T} \text{ in } X^* \text{ such that } \{y_n, y^*_n\}_{n=1}^{Q} \cup \{x_n, x^*_n\}_{n=1}^{T} \text{ is biorthogonal} \\ \text{with } \|x^*_n\| < H \text{ for } 1 \le n \le T \text{ and } U = X \cap \Big\{ \bigcap_{n=1}^{T} x^*_{n\perp} \Big\} \end{gathered} \tag{2.12}$$

($V = V_{T''} + V_{0,T''} = \mathrm{span}\{x'_n\}_{n=1}^{T'} + Q'_{\perp} \cap V_{T''} + V_{0,T''} = X' + \mathrm{span}\{x''_n\}_{n=1}^{T'''} + Q'_{\perp} \cap V_{T''} + V_{0,T''} = X' + W$). Indeed, otherwise, for each R there would exist a subsequence of $T' - T + 1$ elements of $\{n\}_{n=1}^{T'}$, which, by means of a rearrangement, we can call $\{n\}_{n=1}^{T'-T+1}$, such that, for each n with $1 \le n \le T' - T + 1$,

$$\begin{gathered} \text{there exist } u_n \in Y, u'_n \in \mathrm{span}\{x'_k\}_{k(\ne n)=1}^{T'}, u''_n \in Q'_{\perp} \cap V_{T''} \\ \text{and } u'''_n \in V_{0,T''}, \text{ so that } \|x'_n + u_n + u'_n + u''_n + u'''_n\| \le 1/H. \end{gathered} \tag{2.13}$$

But (2.13) would imply, for a sufficiently large R, the existence of n' and n'', with $1 \le n', n'' \le T' - T + 1$, such that

$$\begin{gathered} u'_{n'} = a_{n'} x'_{n''} + v_{n'} \text{ and } u'_{n''} = a_{n''} x'_{n'} + v_{n''}, \\ \text{with span } \{x'_k\}_{k(\ne n', n'')=1}^{T'} \supset \{v_{n'}, v_{n''}\} \text{ and } \max\{|a_{n'}|, |a_{n''}|\} < \tfrac{1}{4}, \\ \text{moreover with } \|u_{n'} - u_{n''}\| < 1/2H \end{gathered} \tag{2.14}$$

(indeed by (2.13), for each n with $1 \le n \le T' - T + 1$, setting $u'_n = \sum_{k(\ne n)=1}^{T'} a_{n,k} x'_k$, it follows that $\{1 + \sum_{k(\ne n)=1}^{T'} (a_{n,k})^2\}^{1/2} < 2\|x'_n + u'_n\|$ (since by (2.10) and (2.8), $\{x'_n\}_{n=1}^{T'}$ is $(1 + 1/2^{T''})$-equivalent to the natural basis of l_2^T) $< 4d\|x'_n + u'_n + u''_n\|$ (by (2.11)) $< (1/d)\|x'_n + u_n + u'_n + u''_n + u'''_n\|$ (by (2.8)) $\le 1/dH$ (by (2.13)); therefore, since in (2.8) d does not depend on k (hence on T'), if R is sufficiently large (hence $T' - T + 1$ is sufficiently large), it would be easy to verify the existence of a subsequence $\{n_i\}_{i=1}^{T''''}$ of $\{n\}_{n=1}^{T'-T+1}$ such that

$|a_{n_i,n_j}| < \frac{1}{4}$ for each $1 \le i \ne j \le T''''$; finally by (2.13) and by the proof of (2.8) it would follow that

$$\|u_n\| \le \frac{1}{d'}\|x'_n + u_n + u'_n + u''_n + u'''_n\| \le \frac{1}{d'H} \quad \text{for } 1 \le n \le T' - T + 1,$$

hence $\|u_{n_i}\| < 1/d'H$ for $1 \le i \le T''''$, where T'''' increases with R; therefore, if R (hence T'''') is sufficiently large, there would exist n' and n'' of $\{n_i\}_{i=1}^{T''''}$ such that $\|u_{n'} - u_{n''}\| < 1/2H$; this completes the proof of (2.14)). From (2.13) and (2.14) it would follow that

$$\begin{aligned} \|(x'_{n'} - x'_{n''}) &+ (a_{n'}x'_{n''} - a_{n''}x'_{n'}) + (v_{n'} - v_{n''}) \\ &+ (u''_{n'} - u''_{n''}) + u'''_{n'} - u'''_{n''})\| \\ &< \frac{2}{H} + \frac{1}{2H}. \end{aligned} \tag{2.15}$$

On the other hand, by (2.13), (2.8), (2.11), (2.10) and (2.14), it also would follow that

$$\begin{aligned} &\|(x'_{n'} - x'_{n''}) + (a_{n'}x'_{n''} - a_{n''}x'_{n'}) + (v_{n'} - v_{n''}) + (u''_{n'} - u''_{n''}) + (u'''_{n'} - u'''_{n''})\| \\ &\quad > (8/H)\|(x'_{n'} - x'_{n''}) + (a_{n'}x'_{n''} - a_{n''}x'_{n'}) + (v_{n'} - v_{n''}) + (u''_{n'} - u''_{n''})\| \\ &\quad > (4/H)\|(x'_{n'} - x'_{n''}) + (a_{n'}x'_{n''} - a_{n''}x'_{n'}) + (v_{n'} - v_{n''})\| \\ &\quad \ge (4/H)\|x'_{n'} - a_{n''}x'_{n'}\| > 3/H, \end{aligned}$$

which would contradict (2.15); this means that (2.13) cannot be true for each R, hence there exists a sufficiently large R for which (2.12) is true.

We go on to verify the last relation of the assertion. By (2.12), (2.8) and (2.11) we have that:

$$\begin{aligned} \operatorname{dist}(x, U) &= \operatorname{dist}(x, Y + \operatorname{span}\{x''_n\}_{n=1}^{T'''} + Q'_\perp \cap V_{T''} + V_{0,T''}) \\ &\ge 4d \cdot \operatorname{dist}(x, \operatorname{span}\{x''_n\}_{n=1}^{T'''} + Q'_\perp \cap V_{T''}) \\ &\ge 2d \cdot \operatorname{dist}(x, \operatorname{span}\{x''_n\}_{n=1}^{T'''}); \end{aligned}$$

hence again by (2.12), since by (2.10) and (2.8) $\{x'_n\}_{n=1}^{T'}$ is $(1+1/2^{T''})$-equivalent to the natural basis of $l_1^{T'}$, we can conclude that $\operatorname{dist}(x, U) \ge d\|x\|$. This, by (2.9) and (2.12), completes the proof of Lemma 2.4.

At this point we are able to state the following analogous general property.

LEMMA 2.5: *For every Banach space X there exists an integer $H \geq 3$ such that, for each positive integer m and for each biorthogonal system $\{y_n, y_n^*\}_{n=1}^Q$ of X, there exist $\{x_{m,n}\}_{n=1}^{T_m}$ in X and $\{x_{m,n}^*\}_{n=1}^{T_m}$ in X^* such that*

$$\{y_{n'}y_n^*\}_{n=1}^Q \cup \{x_{m,n}, x_{m,n}^*\}_{n=1}^{T_m} \text{ is biorthogonal with } \|x_{m,n}\| = 1 \text{ and}$$
$$\|x_{m,n}^*\| < H \text{ for } 1 \leq n \leq T_m; \text{ moreover }, \{x_{m,n}, x_{m,n}^*\}_{n=1}^{T_m} =$$
$$\{\{u_{m,j,n}, u_{m,j,n}^*\}_{n=1}^{2^m}\}_{j=1}^{R_m} \text{ such that, for each } x_0(\neq 0) \text{ of } X,$$
$$\text{there exists } j(0,m), \text{ with } 1 \leq j(0,m) \leq R_m, \text{ such that}$$
$$\sum_{n=1}^{2^m} |u_{m,j(0,m),n}^*(x_0)| < \|x_0\|/2^m.$$

Proof: If X has type > 1, by Lemma 2.4 there exists a positive integer H such that, for each positive integer m, the statement of Lemma 2.4 holds, where we can set $T = T_m = 2^m R_m$, hence

$$\{x_n, x_n^*\}_{n=1}^T = \{x_{m,n}, x_{m,n}^*\}_{n=1}^{T_m} = \{\{u_{m,j,n}, u_{m,j,n}^*\}_{n=1}^{2^m}\}_{j=1}^{R_m},$$

where now $\{R_m\}^{1/2} = \{T_m/2^m\}^{1/2} \geq 2^{2m+1}/d$. Therefore the first part of our thesis is satisfied; moreover, for each $x_0(\neq 0)$ of X and for each m, it is sufficient to verify that there exists $j(0,m)$, with $1 \leq j(0,m) \leq R_m$, such that

$$|u_{m,j(0,m),n}^*(x_0)| < \frac{\|x_0\|}{2^{2m}} \quad \text{for } 1 \leq n \leq 2^m. \tag{2.16}$$

Indeed, otherwise, there would exist, for each j with $1 \leq j \leq R_m, n(0,j)$ such that $|u_{m,j,n(0,j)}^*(x_0)| \geq \|x_0\|/2^{2m}$; therefore, since $\{\{u_{m,j,n}\}_{n=1}^{2^m}\}_{j=1}^{R_m}$ is $(1+1/2^m)$-equivalent to the natural basis of $l_2^{T_m}$, and since, moreover, by the notations of Lemma 2.4, $\{x_0 - \sum_{n=1}^{T_m} x_{m,n}^*(x_0)x_{m,n}^*\} \in U$, where $U = X \cap \{\bigcap_{n=1}^T x_{n\perp}^*\}$, it would follow that

$$\begin{aligned}
\|x_0\| &\geq \operatorname{dist}\Big(\sum_{j=1}^{R_m}\sum_{n=1}^{2^m} u_{m,j,n}^*(x_0)u_{m,j,n}, U\Big) \geq d\Big\|\sum_{j=1}^{R_m}\sum_{n=1}^{2^m} u_{m,j,n}^*(x_0)u_{m,j,n}\Big\| \\
&> \frac{d}{2}\Big\{\sum_{j=1}^{R_m}\sum_{n=1}^{2^m}[u_{m,j,n}^*(x_0)]^2\Big\}^{1/2} \geq \frac{d}{2}\Big\{\sum_{j=1}^{R_m}[u_{m,j,n(0,j)}^*(x_0)]^2\Big\}^{1/2} \\
&\geq \frac{d}{2}\Big\{\frac{\|x_0\|}{2^{2m}}(R_m)^{1/2}\Big\} \geq \frac{d}{2}\Big\{\frac{\|x_0\|}{2^{2m}}\Big\}\frac{2^{2m+1}}{d} = \|x_0\|.
\end{aligned}$$

We now pass to the case where l_1 is finitely represented in X. For a fixed positive integer m, we use the statement of Lemma 2.3 for $\varepsilon < 1$, hence, again, we have that

$$\{x_n, x_n^*\}_{n=1}^{T} = \{x_{m,n}, x_{m,n}^*\}_{n=1}^{T_m} = \{\{u_{m,j,n}, u_{m,j,n}^*\}_{n=1}^{2^m}\}_{j=1}^{R_m}$$
$$\text{where } T = T_m = 2^{M(m)} = 2^m R_m \text{ with } R_m = 2^{M(m)-m}.$$

Again, for each $x_0(\neq 0)$ of X, it is sufficient to see that there exists $j(0,m)$ with $1 \leq j(0,m) \leq R_m$ such (2.16) holds: Indeed, otherwise there would exist, for each j with $1 \leq j \leq R_m, n(0,j)$ such that $|u_{m,j,n(0,j)}^*(x_0)| \geq \|x_0\|/2^{2m}$, therefore by the statement of Lemma 2.3, since $\{x_0 - \sum_{n=1}^{T_m} x_{m,n}^*(x_0)x_{m,n}^*\} \in U$, where again $U = X \cap \{\bigcap_{n=1}^{T} x_{n\perp}^*\}$, it would follow that

$$\begin{aligned}
\|x_0\| &\geq \operatorname{dist}\Big(\sum_{n=1}^{2^{M(m)}} x_{m,n}^*(x_0)x_{m,n}, U\Big) = \operatorname{dist}\Big(\sum_{j=1}^{R_m}\sum_{n=1}^{2^m} u_{m,j,n}^*(x_0)u_{m,j,n}, U\Big) \\
&\geq 2^{1+3m-M(m)} \sum_{j=1}^{R_m}\sum_{n=1}^{2^m} |u_{m,j,n}^*(x_0)| \geq 2^{1+3m-M(m)} \sum_{j=1}^{R_m} |u_{m,j,n(0,j)}^*(x_0)| \\
&\geq 2^{1+3m-M(m)}\{R_m\}\frac{\|x_0\|}{2^{2m}} = 2^{1+3m-M(m)}\{2^{M(m)-m}\}\frac{\|x_0\|}{2^{2m}} = 2\|x_0\|.
\end{aligned}$$

This completes the proof of Lemma 2.5.

Remark 2.1: From the proof of Lemma 2.5 we have that, for each sequence $\{\{a_{j,n}\}_{n=1}^{2^m}\}_{j=1}^{R_m}$ of numbers and for each positive number a, if, for no $j(0,m)$, $\sum_{n=1}^{2^m} |a_{j(0,m),n}| < a/2^m$, then it follows that

$$\operatorname{dist}\Big(\sum_{j=1}^{R_m}\sum_{n=1}^{2^m} a_{j,n}u_{m,j,n}, U\Big) > a, \qquad \text{where } U = X \cap \Big\{\bigcap_{n=1}^{T} x_{n\perp}^*\Big\}.$$

We shall use this remark in §4, in the proof of (b) of Proposition 4.1.

Finally, we can state the existence of an M-basis with controlled coefficients.

PROPOSITION 2.1: *Every separable Banach space X has a norming M-basis $\{x_n\}$ with $\{x_n, x_n^*\}$ biorthogonal, $\|x_n\| = 1$ and $\|x_n^*\| < H$ for each n, where H is an integer ≥ 3, with the following property:*

there is a partition

$$\{x_n, x_n^*\} = \{\{\{u_{m,j,n}, u_{m,j,n}^*\}_{n=1}^{2^m}\}_{j=1}^{R_m} \cup \{v_{m,n}, v_{m,n}^*\}_{n=1}^{P_m}\}_{m=1}^{\infty}$$

(2.17) *such that, for each m and for each $x_0(\neq 0)$ of X, there exists $j(0,m)$, with $1 \leq j(0,m) \leq R_m$, such that*

$$\sum_{n=1}^{2^m} |u_{m,j(0,m),n}^*(x_0)| < \|x_0\|/2^m.$$

Proof: Let $\{w_n\}$ be a norming M-basis of X with $\{w_n, w_n^*\}$ biorthogonal. Fix an integer $m > 1$ and assume to have already defined

(2.18)
$$\begin{aligned}
&\{\{u_{k,n}, u_{k,n}^*\}_{n=1}^{S_k} \cup \{v_{k,n}, v_{k,n}^*\}_{n=1}^{P_k}\}_{k=1}^{m-1} \text{ biorthogonal where, for each}\\
&k \text{ with } 1 \le k \le m-1, \{u_{k,n}, u_{k,n}^*\}_{n=1}^{S_k} = \{\{u_{k,j,n}, u_{k,j,n}^*\}_{n=1}^{2^k}\}_{j=1}^{R_k};\\
&\|u_{k,n}\| = 1 \text{ and } \|u_{k,n}^*\| < H \text{ for } 1 \le n \le S_k,\\
&\|v_{k,n}\| = 1 \text{ and } \|v_{k,n}^*\| < 3 \text{ for } 1 \le n \le P_k;\\
&\text{the second part of (2.17) for } m \text{ replaced by } k \text{ is verified;}\\
&\text{moreover, if } X_k = \operatorname{span}\{\{u_{j,n}\}_{n=1}^{S_j} \cup \{v_{j,n}\}_{n=1}^{P_j}\}_{j=1}^{k},\\
&W_k = X \cap \Big\{\bigcap_{j=1}^{k}\Big\{\Big\{\bigcap_{n=1}^{S_j} u_{j,n\perp}^*\Big\} \cap \Big\{\bigcap_{n=1}^{P_j} v_{j,n\perp}^*\Big\}\Big\}\Big\} \text{ and}\\
&B^*(k) = \{\{u_{j,n}^*\}_{n=1}^{S_j} \cup \{v_{j,n}^*\}_{n=1}^{P_j}\}_{j=1}^{k}, \text{ we have that } X = X_k + W_k,\\
&\operatorname{dist}(w_j, X_k) < 1/2^k \text{ for } 1 \le j \le k \text{ and span } \{B^*(k)\} \supset \{w_j^*\}_{j=1}^{k}.
\end{aligned}$$

We are then going to define $\{u_{m,n}, u_{m,n}^*\}_{n=1}^{S_m} \cup \{v_{m,n}, v_{m,n}^*\}_{n=1}^{P_m}$ such that (2.18) is verified also if $m-1$ is replaced by m. This is an inductive procedure since the construction of the first step (that is (2.18) for $m = 1$) is only an obvious simplified version of the construction of the general step.

Note that the construction of $\{u_{m,n}, u_{m,n}^*\}_{n=1}^{S_m}$ with $\|u_{m,n}\| = 1$ and $\|u_{m,n}^*\| < H$ for $1 \le n \le S_m$, such that the second part of (2.17) is verified, follows directly from Lemma 2.5. Consequently $\{y_n, y_n^*\}_{n=1}^{Q}$ of Lemma 2.5 is now replaced by $\{\{u_{k,n}, u_{k,n}^*\}_{n=1}^{S_k} \cup \{v_{k,n}, v_{k,n}^*\}_{n=1}^{P_k}\}_{k=1}^{m-1}$ and we proceed to define $\{v_{m,n}, v_{m,n}^*\}_{n=1}^{P_m}$. First there exist $\{v'_{m,n}\}_{n=1}^{P'_m}$ in $W'_m = W_{m-1} \cap \{\bigcap_{n=1}^{S_m} u_{m,n\perp}^*\}$ and $\{v'^*_{m,n}\}_{n=1}^{P'_m}$ in X^* with $\|v'_{m,n}\| = 1$ for $1 \le n \le P'_m$, $\{v'_{m,n}, v'^*_{m,n}\}_{n=1}^{P'_m}$ biorthogonal and $X \cap \{\bigcap_{n=1}^{P'_m} v'^*_{m,n\perp}\} \supset X_{m-1} + \operatorname{span}\{u_{m,n}\}_{n=1}^{S_m}$, such that, for $1 \le n \le m$,

$$\operatorname{dist}(w_n, X_{m-1} + \operatorname{span}\{u_{m,k}\}_{k=1}^{S_m} + \operatorname{span}\{v'_{m,n}\}_{k=1}^{P'_m}) < 1/2^m.$$

If w_m^* is an element of span $\{B^*(m-1) \cup \{y_{m,n}^*\}_{n=1}^{S_m} \cup \{v'^*_{m,n}\}_{n=1}^{P'_m}\}$ we set $P''_m = P'_m$; otherwise we set $P''_m = P'_m + 1$ and (by [60], p. 39, th. 3) there exists $v'_{m,P''_m} \in W''_m = W'_m \cap \{\bigcap_{n=1}^{P'_m} v'^*_{m,n\perp}\}$ with $\|v'_{m,P''_m}\| = 1$ and $w_m^*(v'_{m,P''_m}) \neq 0$. In this case we set

$$\begin{aligned}
v'^*_{m,P''_m} = \Big\{ w_m^* - \sum_{k=1}^{m-1} \Big[\sum_{n=1}^{S_k} w_m^*(u_{k,n}) u_{k,n}^* + \sum_{n=1}^{P_k} w_m^*(v_{k,n}) v_{k,n}^*\Big] \\
- \sum_{n=1}^{S_m} w_m^*(u_{m,n}) u_{m,n}^* - \sum_{n=1}^{P'_m} w_m^*(v'_{m,n}) v'^*_{m,n} \Big\} / w_m^*(v'_{m,P''_m}).
\end{aligned}$$

At this point, by Reference III* of the Introduction, there exist

$$\{v'_{m,n}\}_{n=P''_m+1}^{P_m} \text{ in } W''_m \cap v'^*_{m,P''_m\perp} \text{ and } \{v'^*_{m,n}\}_{n=P''_m+1}^{P_m} \text{ in } X^*$$
$$\text{with } \{v'_{m,n}, v'^*_{m,n}\}_{n=P''_m+1}^{P_m} \text{ biorthogonal and}$$
$$X \cap \left\{ \bigcap_{n=P''_m+1}^{P_m} v'^*_{m,n\perp} \right\} \supset X_{m-1} + \operatorname{span}\{u_{m,n}\}_{n=1}^{S_m} + \operatorname{span}\{v'_{m,n}\}_{n=1}^{P''_m},$$
$$\text{such that } \|v'_{m,n}\| = 1 = \|v'^*_{m,n}\| \text{ for } P''_m + 1 \leq n \leq P_m,$$
$$\max\{\|v'^*_{m,n}\| : 1 \leq n \leq P''_m\} \quad \text{and} \quad P_m - P''_m \geq P''_m 4.$$

Therefore, by Reference II* of the Introduction, we pass from

$$\{v'_{m,n}, v'^*_{m,n}\}_{n=1}^{P_m} \text{ to } \{v_{m,n}, v^*_{m,n}\}_{n=1}^{P_m} \text{ with span } \{v_{m,n}\}_{n=1}^{P_m}$$
$$= \operatorname{span}\{v'_{m,n}\}_{n=1}^{P_m}, \operatorname{span}\{v^*_{m,n}\}_{n=1}^{P_m} = \operatorname{span}\{v'^*_{m,n}\}_{n=1}^{P_m},$$
$$\|v_{m,n}\| = 1 \text{ and } \|v^*_{m,n}\| < 3 \text{ for } 1 \leq n \leq P_m.$$

Setting now $X_m = X_{m-1} + \operatorname{span}\{u_{m,n}\}_{n=1}^{S_m} + \operatorname{span}\{v_{m,n}\}_{n=1}^{P_m}, W_m = W'_m \cap \{\bigcap_{n=1}^{P_m} v^*_{m,n\perp}\}$ and $B^*(m) = B^*(m-1) \cup \{u^*_{m,n}\}_{n=1}^{S_m} \cup \{v^*_{m,n}\}_{n=1}^{P_m}$, we obtain (2.18) for m instead of $m-1$.

Proceeding in this way we define $\{x_n\}$ with $\sum_{m=1}^{\infty} X_m$ dense in $\operatorname{span}\{w_n\}$, hence in X; on the other hand $\operatorname{span}\{x_n^*\} \supset \{w_n^*\}$, therefore $\{x_n\}$ is a norming M-basis of X; this completes the proof of Proposition 2.1.

3. Existence of the uniformly minimal basis with quasi-fixed brackets and permutations

Proof of Theorem I: Starting from the norming M-basis $\{x_n\}$ of Proposition 2.1 we shall define a new sequence $\{y_n\}$, with $\{y_n, y_n^*\}$ biorthogonal, by means of the sequence $\{r_m\}$ of Reference I* of the Introduction and by means of a subsequence $\{t(m)\}$ of $\{m\}$ such that, for each m,

$$\text{setting } q_m = r_{t(m)}, \text{ we have that span } \{y_n\}_{n=q_m+1}^{q_{m+1}} =$$
$$\operatorname{span}\{x_n\}_{n=q_m+1}^{q_{m+1}} \text{ and span } \{y_n^*\}_{n=q_m+1}^{q_{m+1}} = \operatorname{span}\{x_n^*\}_{n=q_m+1}^{q_{m+1}}, \tag{3.1}$$
$$\text{moreover } \|y_n\| = 1 \text{ and } \|y_n^*\| < 3H \text{ for } q_m + 1 \leq n \leq q_{m+1}.$$

Setting $\{y_n, y_n^*\}_{n=1}^{q_0} = \{x_n, x_n^*\}_{n=1}^{r_1}$, for $m \geq 0$ we proceed by induction: We suppose we have defined $\{y_n, y_n^*\}_{n=1}^{q_m}$ and we are going to define $\{y_n, y_n^*\}_{n=q_m+1}^{q_{m+1}}$.

We shall define an integer $t(m+1)$ and a partition

$$\{x_n\}_{n=q_m+1}^{q_{m+1}} = \{x_n\}_{n=r_{t(m)}+1}^{r_{t(m)+1}} \cup \{x_n\}_{n=r_{t(m)+1}+1}^{r_{t(m+1)}},$$

$$\text{with } \{x_n\}_{n=r_{t(m)+1}+1}^{r_{t(m+1)}} = \bigcup_{i=1}^{r_{t(m)+1}-r_{t(m)}} \{x_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}$$

(where $t(m,0) = t(m)+1$, hence $t(m+1)$ is $t(m,i)$ for $i = r_{t(m)+1} - r_{t(m)}$), so that, writing

$$\{x_n\}_{n=q_m+1}^{q_{m+1}} = \{x_{q_m+i} \cup \{x_n\}_{n=r_{t(m,i-1)}+1}^{t_{t(m,i)}}\}_{i=1}^{r_{t(m)+1}-r_{t(m)}}, \tag{3.2}$$

we shall define $\{y_n, y_n^*\}_{n=q_m+1}^{q_{m+1}}$ biorthogonal such that, for each i with $1 \le i \le r_{t(m)+1} - r_{t(m)}$,

$$\operatorname{span}\{y_{q_m+i} \cup \{y_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}\} = \operatorname{span}\{x_{q_m+i} \cup \{x_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}\}$$

$$\text{and span } \{y_{q_m+i}^* \cup \{y_n^*\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}\} = \operatorname{span}\{x_{q_m+i}^* \cup \{x_n^*\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}\}.$$

Again we shall proceed by induction: Fix an integer i with $2 \le i \le r_{t(m)+1} - r_{t(m)}$ and suppose we have already defined the biorthogonal sequence $\{y_{q_m+j}, y_{q_m+j}^*\}_{j=1}^{i-1} \cup \{y_n, y_n^*\}_{n=r_{t(m)+1}+1}^{r_{t(m,i-1)}}$ such that, for each j with $1 \le j \le i-1$, the second part of (3.2) is verified; then we are going to define $\{y_{q_m+i}, y_{q_m+i}^*\} \cup \{y_n, y_n^*\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}$ (this is a valid induction procedure because the construction for $i=1$ is only an obvious simplified version of the construction for the general $i > 1$). Setting

$$S_{m,i} = 2^{m+6} H r_{t(m,i-1)} \text{ we select a sequence } \{w_{m,i,s}\}_{s=1}^{L_{m,i}}$$

$$\text{with } w_{m,i,s} = \sum_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}} a_{m,i,s,n} x_n \text{ for } 1 \le s \le L_{m,i} \text{ which is} \tag{3.3}$$

$(1/S_{m,i})$-dense in the ball of radius $S_{m,i}$ of span $\{x_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}$.

We can choose two positive integers, $A(m,i)$ and $t(m,i)$, as follows (we use the sequences of Proposition 2.1):

$A(m,i)$ is the first integer $\ge 8HS_{m,i}^3$ such that there exists another integer

$t(m,i)$ for which

(3.4)
$$\begin{aligned}&\{x_n\}_{n=r_{t(m,i-1)+1}+1}^{r_{t(m,i)}} = \{x_{m,i,n}\}_{n=1}^{T_{m,i}} \cup \{x_{m,i,0,n}\}_{n=1}^{T_{m,i,0}},\\ &\text{where } \{x_{m,i,n}\}_{n=1}^{T_{m,i}} = \{\{\{u_{f,j,h}\}_{h=1}^{2^f}\}_{j=1}^{R_f}\}_{f=A(m,i)+1}^{A(m,i)+L_{m,i}S_{m,i}^2},\\ &\text{which we rewrite by means of the following partition :}\\ &\{\{\{\{u_{f(m,i,s,k),j,h}\}_{h=1}^{2^{f(m,i,s,k)}}\}_{j=1}^{R_{f(m,i,s,k)}}\}_{k=1}^{S_{m,i}^2}\}_{s=1}^{L_{m,i}}, \text{ and moreover}\\ &\text{such that } T_{m,i,0} \geq [1 + r_{t(m,i-1)+1} - r_{t(m,i-1)}]4^{4H^2S_{m,i}^3T_{m,i}}\end{aligned}$$

(we point out that $T_{m,i}$ is determined by $S_{m,i}$ (hence by $A(m,i)$) and by $L_{m,i}$, while $T_{m,i,0}$ (hence $r_{t(m,i)}$) is determined by $S_{m,i}$ and $T_{m,i}$). Set

(3.5)
$$y'_{q_m+i} = \frac{x_{q_m+i}}{S_{m,i}^2} - \sum_{s=1}^{L_{m,i}} \sum_{k=1}^{S_{m,i}^2} S_{m,i}^{k/2} \sum_{j=1}^{R_{f(m,i,s,k)}} u_{f(m,i,s,k),j,1}$$
$$\text{and } y'^*_{q_m+i} = S_{m,i}^2 x^*_{q_m+i}$$

(hence $y'^*_{q_m+i}(y'_{q_m+i}) = 1$ and $S_{m,i}^{k/2} y'^*_{q_m+i} = kx^*_{q_m+i}$ for $1 \leq k \leq S_{m,i}^2$). Moreover, for each fixed set of indices s,k,j, with $1 \leq s \leq L_{m,i}, 1 \leq k \leq S_{m,i}^2$ and $1 \leq j \leq R_{f(m,i,s,k)}$, we set

(3.6)
$$\begin{aligned}&y'_{f(m,i,s,k),j,1} = u_{f(m,i,s,k),j,1} + w_{m,i,s} \text{ and } y'^*_{f(m,i,s,k),j,1}\\ &= u^*_{f(m,i,s,k),j,1} + kx^*_{q_m+i}, \text{ while for } 2 \leq h \leq 2^{f(m,i,s,k)} \text{ we set}\\ &y'_{f(m,i,s,k),j,h} = u_{f(m,i,s,k),j,h} \text{ and } y'^*_{f(m,i,s,k),j,h} = u^*_{f(m,i,s,k),j,h}\end{aligned}$$

(hence, by (3.4) and (3.5), $\{y'_{q_m+i}, y'^*_{q_m+i}\} \cup \{y'_{m,i,n}, y'^*_{m,i,n}\}_{n=1}^{T_{m,i}}$ is biorthogonal). We set also

(3.7)
$$\begin{aligned}&y'_{m,i,0,n} = x_{m,i,0,n} \text{ and } y'^*_{m,i,0,n} = x^*_{m,i,0,n} \text{ for } 1 \leq n \leq T_{m,i,0},\\ &y'_n = x_n \text{ for } r_{t(m,i-1)} + 1 \leq n \leq r_{t(m,i-1)+1}.\end{aligned}$$

By (3.3) and (3.6), setting for $r_{t(m,i-1)} + 1 \leq n \leq r_{t(m,i-1)+1}$

(3.8) $y'^*_n = x^*_n - \sum_{s=1}^{L_{m,i}} \sum_{k=1}^{S_{m,i}^2} \sum_{j=1}^{R_{f(m,i,s,k)}} a_{m,i,s,n} y'^*_{f(m,i,s,k),j,1}$, we have that

$$\{y'_{q_m+i}, y'^*_{q_m+i}\} \cup \{y'_n, y'^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}$$

is biorthogonal, with

$$\begin{aligned}&\operatorname{span}\{y'_{q_m+i} \cup \{\{\{y'_{f(m,i,s,k),j,1}\}_{j=1}^{R_{f(m,i,s,k)}}\}_{k=1}^{S_{m,i}^2}\}_{s=1}^{L_{m,i}} \cup \{y'_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}\}\\ &= \operatorname{span}\{x_{q_m+i} \cup \{\{\{u_{f(m,i,s,k),j,1}\}_{j=1}^{R_{f(m,i,s,k)}}\}_{k=1}^{S_{m,i}^2}\}_{s=1}^{L_{m,i}} \cup \{x_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}\}\end{aligned}$$

and with

$$\begin{aligned}&\operatorname{span}\{y'^*_{q_m+i}\cup\{\{\{y'^*_{f(m,i,s,k),j,1}\}_{j=1}^{R_{f(m,i,s,k)}}\}_{k=1}^{S^2_{m,i}}\}_{s=1}^{L_{m,i}}\\&\cup\{y'^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}\}\\=&\operatorname{span}\{x^*_{q_m+i}\cup\{\{\{u^*_{f(m,i,s,k),j,1}\}_{j=1}^{R_{f(m,i,s,k)}}\}_{k=1}^{S^2_{m,i}}\}_{s=1}^{L_{m,i}}\\&\cup\{x^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}\}.\end{aligned}$$

Using (3.4), (3.5) and Proposition 2.1, we have that $\|y'_{q_m+i}\| < 2T_{m,i}$ and $\|y'^*_{q_m+i}\| < HS^2_{m,i}$, that is $\|y'_{q_m+i}\|\cdot\|y'^*_{q_m+i}\| < 2HS^2_{m,i}T_{m,i}$; moreover by (3.6) and (3.3), for each set of indices $\{s,k,j\}$, with $1\le s\le L_{m,i}, 1\le k\le S^2_{m,i}$ and $1\le j\le R_{f(m,i,s,k)}$, we have that $\|y'_{f(m,i,s,k),j,1}\| < 2S_{m,i}$ and $\|y'^*_{f(m,i,s,k),j,1}\| < 2HS^2_{m,i}$, that is $\|y'_{f(m,i,s,k),j,1}\|\cdot\|y'^*_{f(m,i,s,k),j,1}\| < 4HS^3_{m,i}$; moreover, for a fixed n with $r_{t(m,i-1)}+1\le n\le r_{t(m,i-1)+1}$, since in (3.3) by Proposition 2.1, $|a_{m,i,s,n}| < HS_{m,i}$ for $1\le s\le L_{m,i}$, by the first part of (3.8) and by what we have just found above, we have that $\|y'^*_n\| < \|x^*_n\| + \max\{|a_{m,i,s,n}| : 1\le s\le L_{m,i}\}\cdot\max\{\|y'^*_{f(m,i,s,k),j,1}\| : 1\le s\le L_{m,i}, 1\le k\le S^2_{m,i}$ and $1\le j\le R_{f(m,i,s,k)}\}T_{m,i} < H + HS_{m,i}\cdot 2HS^2_{m,i}\cdot T_{m,i}$, that is $\|y'_n\|\cdot\|y'^*_n\| < 4H^2S^3_{m,i}T_{m,i}$.

Therefore by the last relation of (3.4), by Reference II* of the Introduction and by what we have just found above, there exists a biorthogonal system

(3.9)
$$\{y_n,y^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}\cup\{y_{q_m+i},y^*_{q_m+i}\}\cup\{y_{m,i,0,n},y^*_{m,i,0,n}\}_{n=1}^{T_{m,i,0}}$$

with

$$\begin{aligned}&\|y_n\|=1 \text{ and } \|y^*_n\|<3H \text{ for } r_{t(m,i-1)}+1\le n\le r_{t(m,i-1)+1},\\&\|y_{q_m+i}\|=1 \text{ and } \|y^*_{q_m+i}\|<3H, \|y_{m,i,0,n}\|=1 \text{ and } \|y^*_{m,i,0,n}\|<3H\\&\text{for } 1\le n\le T_{m,i,0};\end{aligned}$$

moreover with

$$\begin{aligned}&\operatorname{span}\{\{y_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}\cup y_{q_m+i}\cup\{y_{m,i,0,n}\}_{n=1}^{T_{m,i,0}}\}\\=&\operatorname{span}\{\{y'_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}\cup y'_{q_m+i}\cup\{y'_{m,i,0,n}\}_{n=1}^{T_{m,i,0}}\}\end{aligned}$$

and

$$\begin{aligned}&\operatorname{span}\{\{y^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}\cup y^*_{q_m+i}\cup\{y^*_{m,i,0,n}\}_{n=1}^{T_{m,i,0}}\}\\=&\operatorname{span}\{\{y'^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}}\cup y'^*_{q_m+i}\cup\{iy'^*_{m,i,0,n}\}_{n=1}^{T_{m,i,0}}\}.\end{aligned}$$

By the same procedure, by Proposition 2.1, by (3.6) and by the first part of (3.4), for each set of indices $\{s,k,j\}$, with $1\le s\le L_{m,i}, 1\le k\le S^2_{m,i}$ and

$1 \leq j \leq R_{f(m,i,s,k)}$, since $f(m,i,s,k) > A(m,i) \geq 8HS^3_{m,i}$ and $\|y'_{f(m,i,s,k),j,1}\| \cdot \|y'^*_{f(mi,s,k),j,1}\| < 4HS^3_{m,i}$, we can obtain a biorthogonal system

$$(3.10)\quad \begin{aligned} &\{y_{f(m,i,s,k),j,h}, y^*_{f(m,i,s,k),j,h}\}_{h=1}^{2^{f(m,i,s,k)}} \text{ with } \|y_{f(m,i,s,k),j,h}\| = 1 \\ &\text{and } \|y^*_{f((m,i,s,k),j,h)}\| < 3H \text{ for } 1 \leq h \leq 2^{f(m,i,s,k)}, \\ &\operatorname{span}\{y_{f(m,i,s,k),j,h}\}_{h=1}^{2^{f(m,i,s,k)}} = \operatorname{span}\{y'_{f(m,i,s,k),j,h}\}_{h=1}^{2^{f(m,i,s,k)}} \text{ and} \\ &\operatorname{span}\{y^*_{f(m,i,s,k),j,h}\}_{h=1}^{2^{f(m,i,s,k)}} = \operatorname{span}\{y'^*_{f(m,i,s,k),j,h}\}_{h=1}^{2^{f(m,i,s,k)}}. \end{aligned}$$

Therefore, by (3.4)...(3.10), we have that the second part of (3.2) and also the last properties of (3.1) are verified. Proceeding in this way up to $\{y_{q_m+i}, y^*_{q_m+i}\} \cup \{y_n, y^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}$ for $i = r_{t(m)+1} - r_{t(m)}$ and setting $q_{m+1} = r_{t(m,i)}$ for $i = r_{t(m)+1} - r_{t(m)}$, we have that also the first parts of (3.1) and (3.2) are verified. Then $\{y_n\}$ is uniformly minimal and we are going to prove that it is also a basis with quasi-fixed brackets and permutations of X.

Let x_0 be an element of X with $\|x_0\| = 1$, then $\{m\} = \{m(k)\} \cup \{m'(k)\}$ (where one of the subsets may be finite or empty) so that, for each k,

$$\begin{aligned} &|x^*_{q_{m(k)}+i(k)}(x_0)| > 1/S^2_{m(k),i(k)} \quad \text{for some } i(k) \\ &\qquad\qquad \text{with } 1 \leq i(k) \leq r_{t(m(k))+1} - r_{t(m(k))}; \\ &|x^*_{q_{m'(k)}+j}(x_0)| \leq 1/S^2_{m'(k),j} \quad \text{for } 1 \leq j \leq r_{t(m'(k))+1} - r_{t(m'(k))}. \end{aligned}$$

We shall consider separately these two subsequences:

Case (A): Suppose that $\{m(k)\}$ is infinite. Then, in order to simplify notations, we denote this subsequence by $\{m\}$ again and we have that for each m there exists another integer $i(m)$, with $1 \leq i(m) \leq r_{t(m)+1} - r_{t(m)}$, such that

$$(3.11)\quad \begin{aligned} &|x^*_{q_m+i(m)}(x_0)| > 1/S^2_{m,i(m)} \quad \text{while} \\ &|x^*_{q_m+j}(x_0)| \leq 1/S^2_{mj} \quad \text{for } i(m) + 1 \leq j \leq r_{t(m)+1} - r_{t(m)}. \end{aligned}$$

Being with $\{x_n\}$ uniformly minimal, by the remark after (D_2) of the Introduction there exists a subsequence $\{m''\}$ of $\{m\}$ such that, for each m and for each $m' > m''$:

$$(3.12)\quad \begin{aligned} &|x^*_n(x_0)| < \frac{1}{4^{m+2}} \quad \text{for } n > q_{m'}; \quad \text{moreover} \\ &\Big\| \sum_{j=i(m')}^{r_{t(m')+1}-r_{t(m')}} x^*_{q_{m'}+j}(x_0)x_{q_{m'}+j} = \sum_{n=r_{t(m')}+i(m')}^{r_{t(m')+1}} x^*_n(x_0)x_n \Big\| < \frac{1}{2^{m+2}} \end{aligned}$$

(to verify the second inequality note that by the first part

$$\|x^*_{q_{m'}+i(m')}(x_0)x_{q_{m'}+i(m')}\| < \frac{1}{4^{m+2}},$$

moreover by (3.11) and (3.3)

$$\begin{aligned}\Big\|&\sum_{j=i(m')+1}^{r_{t(m')+1}-r_{t(m')}} x^*_{q_{m'}+j}(x_0)x_{q_{m'}+j}\Big\|\\ &< \sum_{j=i(m')+1}^{r_{t(m')+1}-r_{t(m')}} 1/S^2_{m'j}\\ &= \sum_{j=i(m')+1}^{r_{t(m')+1}-r_{t(m')}} 1/(2^{m'+6}Hr_{t(m',j-1)})^2\\ &< \frac{1}{2^{m'+6}} \le \frac{1}{2^{m+6}}).\end{aligned}$$

By (3.1) and (3.2) we have that

$$\begin{aligned}&\sum_{n=1}^{r_{t(m')}+i(m')-1} x^*_n(x_0)x_n + \sum_{n=r_{t(m')+1}+1}^{r_{t(m',i(m')-1)}} x^*_n(x_0)x_n\\ &= \sum_{n=1}^{r_{t(m')}+i(m')-1} y^*_n(x_0)y_n + \sum_{n=r_{t(m')+1}+1}^{r_{t(m',i(m')-1)}} y^*_n(x_0)y_n\end{aligned} \tag{3.13}$$

(elements with indices n such that $r_{t(m')+1}+1 \le n \le r_{t(m',i(m')-1)}$ do not appear in the sum if $i(m')=1$). Moreover there exists another subsequence of $\{m''\}$, which we call again $\{m''\}$, such that for each $m, m'' \ge m+2$ and, for each $m' \ge m''$, there exists $x_{0,m} \in \operatorname{span}\{x_n\}_{n=1}^{r_{t(m',i(m')-1)}}$, with $\|x_0 - x_{0,m}\| < 1/2^{m+4}$. Therefore, by the properties of the sequence $\{r_m\}$ of Reference I* of the Introduction, for each m and for each $m' \ge m''$ we have that:

$$\begin{aligned}&\operatorname{dist}\Big(x_0 - \sum_{n=1}^{r_{t(m',i(m')-1)}} x^*_n(x_0)x_n, \operatorname{span}\{x_n\}_{n=r_{t(m',i(m')-1)}+1}^{r_{t(m',i(m')-1)+1}}\Big)\\ \le&\|x_0 - x_{0,m}\|\\ &+ \operatorname{dist}\Big(x_{0,m} - \sum_{n=1}^{r_{t(m',i(m')-1)}} x^*_n(x_0)x_n, \operatorname{span}\{x_n\}_{n=r_{t(m',i(m')-1)}+1}^{r_{t(m',i(m')-1)+1}}\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq \|x_0 - x_{0,m}\| + \operatorname{dist}\Big(x_{0,m} - \sum_{n=1}^{r_{t(m',i(m')-1)}} x_n^*(x_0)x_n, \operatorname{span}\{x_n\}_{n>r_{t(m',i(m')-1)}}\Big) \\
&\quad + \frac{\Big\|x_{0,m} - \sum_{n=1}^{r_{t(m',i(m')-1)}} x_n^*(x_0)x_n\Big\|}{\Big\{2^{t(m',i(m')-1)}\Big(2+\sum_{n=1}^{r_{t(m',i(m')-1)}} \|x_n^*\|\Big)\Big\}} \\
&\leq 2\|x_0 - x_{0,m}\| + \operatorname{dist}\Big(x_0 - \sum_{n=1}^{r_{t(m',i(m')-1)}} x_n^*(x_0)x_n, \operatorname{span}\{x_n\}_{n>r_{t(m',i(m')-1)}}\Big) \\
&\quad + \Big\|x_{0,m} - \sum_{n=1}^{r_{t(m',i(m')-1)}} x_n^*(x_0)x_n\Big\| \Big\{2^{t(m',i(m')-1)}\Big(2+\sum_{n=1}^{r_{t(m',i(m')-1)}} \|x_n^*\|\Big)\Big\} \\
&= \frac{2\|x_0 - x_{0,m}\| + \Big\|x_{0,m} - \sum_{n=1}^{r_{t(m',i(m')-1)}} x_n^*(x_0)x_n\Big\|}{\Big\{2^{t(m',i(m')-1)}\Big(2+\sum_{n=1}^{r_{t(m',i(m')-1)}} \|x_n^*\|\Big)\Big\}} \\
&< 2\|x_0 - x_{0,m}\| + \frac{\Big\{1+\frac{1}{2^{m+4}}+\sum_{n=1}^{r_{t(m',i(m')-1)}} \|x_n^*\|\Big\}}{\Big\{2^{t(m',i(m')-1)}\Big(2+\sum_{n=1}^{r_{t(m',i(m')-1)}} \|x_n^*\|\Big)\Big\}} \\
&< 2\|x_0 - x_{0,m}\| + \frac{1}{2^{t(m',i(m')-1)}} \\
&< \frac{1}{2^{m+3}} + \frac{1}{2^{t(m')+1}} \\
&< \frac{1}{2^{m+3}} + \frac{1}{2^{m''+1}} \leq \frac{1}{2^{m+2}}.
\end{aligned}
$$

Therefore, by (3.12) and (3.13), for each m and for each $m' \geq m''$, there exists $w_{m'} \in \operatorname{span}\{x_n\}_{n=r_{t(m',i(m')-1)}+1}^{r_{t(m',i(m')-1)+1}}$ such that

$$
\begin{aligned}
&\Big\|x_0 - \Big\{\sum_{n=1}^{r_{t(m')}+i(m')-1} y_n^*(x_0)y_n + \sum_{n=r_{t(m')+1}+1}^{r_{t(m',i(m')-1)}} y_n^*(x_0)y_n + w_{m'}\Big\}\Big\| \\
&= \Big\|x_0 - \Big\{\sum_{n=1}^{r_{t(m',i(m')-1)}} x_n^*(x_0)x_n + w_{m'}\Big\} + \sum_{n=r_{t(m')}+i(m')}^{r_{t(m')+1}} x_n^*(x_0)x_n\Big\| \\
&\leq \Big\|x_0 - \Big\{\sum_{n=1}^{r_{t(m',i(m')-1)}} x_n^*(x_0)x_n + w_{m'}\Big\}\Big\| + \Big\|\sum_{n=r_{t(m')}+i(m')}^{r_{t(m')+1}} x_n^*(x_0)x_n\Big\| \\
&< \frac{1}{2^{m+1}}.
\end{aligned}
\tag{3.14}
$$

We fix m and $m' \geq m''$; our aim is to pick out three indices $k(m'), s(m')$ and $j(m')$, with $1 \leq k(m') \leq S^2_{m',i(m')}, 1 \leq s(m') \leq L_{m',i(m')}$ and $1 \leq j(m') \leq$

$R_{f(m',i(m'),s(m')k(m'))}$, such that

$$\begin{aligned}\Big\| x_0 - \Big\{ &\sum_{n=1}^{r_{t(m')}+i(m')-1} y_n^*(x_0)y_n + \sum_{n=r_{t(m')+1}+1}^{r_{t(m',i(m')-1)}} y_n^*(x_0)y_n \\ &+ \sum_{h=1}^{2^{f(m',i(m'),s(m'),k(m'))}} y^*_{f(m',i(m'),s(m'),k(m')),j(m'),h}(x_0)\cdot \\ &\cdot y_{f(m',i(m'),s(m'),k(m')),j(m'),h}\Big\}\Big\| < \frac{1}{2^m}.\end{aligned} \tag{3.15}$$

Then, since by (3.11) and (3.12)

$$\frac{1}{S^2_{m',i(m')}} < |x^*_{q_{m'}+i(m')}(x_0)| < \frac{1}{4^{m+2}},$$

there exists $k(m')$ with $2^{m+1} \le k(m') \le S^2_{m',i(m')}/2^{m+3}$ such that

$$\Big|\Big\{k(m')|x^*_{q_{m'}+i(m')}(x_0)| - \frac{1}{2^{m+3}}\Big\}\Big| < \frac{1}{4^{m+1}}. \tag{3.16}$$

On the other hand, using (3.14), Proposition 2.1 and (3.3), we have that

$$\|w_{m'}\| < 4Hr_{t(m',i(m')-1)} = \frac{S_{m',i(m')}}{2^{m'+4}} \le \frac{S_{m',i(m')}}{2^{m+4}};$$

therefore, by (3.16), $\|w_{m'}/\{k(m')x^*_{q_{m'}+i(m')}(x_0)\}\| < 2^{m+4}\|w_{m'}\| < S_{m',i(m')}$; hence, by the first part of (3.14) and by (3.3), there exists another integer $s(m')$, with $1 \le s(m') \le L_{m',i(m')}$, such that, by (3.16),

$$\begin{aligned}&\|w_{m'} - k(m')x^*_{q_{m'}+i(m')}(x_0)w_{m',i(m'),s(m')}\| \\ &= k(m')|x^*_{q_{m'}+i(m')}(x_0)| \\ &\quad\cdot \|w_{m'}/\{k(m')x^*_{q_{m'}+i(m')}(x_0)\} - w_{m',i(m'),s(m')}\| \\ &< \frac{k(m')|x^*_{q_{m'}+i(m')}(x_0)|}{S_{m',i(m')}} < \frac{1}{S_{m',i(m')}}.\end{aligned} \tag{3.17}$$

Now we consider the fixed index $f(m',i(m'),s(m'),k(m'))$ and, for each j with

$1 \leq j \leq R_{f(m',i(m'),s(m'),k(m'))}$, by (3.6) and (3.10) we have that

$$\begin{aligned}
&\sum_{h=1}^{2^{f(m',i(m'),s(m'),k(m'))}} y^*_{f(m',i(m'),s(m'),k(m'))j,h}(x_0) y_{f(m',i(m'),s(m'),k(m')),j,h} \\
&= \sum_{h=1}^{2^{f(m',i(m'),s(m'),k(m'))}} y'^*_{f(m',i(m'),s(m'),k(m')),j,h}(x_0) y'_{f(m',i(m'),s(m'),k(m')),j,h} \\
&= \{u^*_{f(m',i(m'),s(m'),k(m')),j,1}(x_0) + k(m') x^*_{q_{m'}+i(m')}(x_0)\} \\
&\quad \cdot \{u_{f(m',i(m'),s(m'),k(m')),j,1} + w_{m',i(m'),s(m')}\} \\
&+ \sum_{h=2}^{2^{f(m',i(m'),s(m'),k(m'))}} u^*_{f(m',i(m'),s(m'),k(m')),j,h}(x_0) u_{f(m',i(m'),s(m'),k(m')),j,h};
\end{aligned}$$

therefore by (2.17) and (3.4) we can choose an integer $j(m')$, with $1 \leq j(m') \leq R_{f(m',i(m'),s(m'),k(m'))}$, such that (since $\|x_0\| = 1$)

$$\begin{aligned}
&\sum_{h=1}^{2^{f(m',i(m'),s(m'),k(m'))}} |u^*_{f(m',i(m'),s(m'),k(m')),j(m'),h}(x_0)| \\
&< 1/2^{f(m',i(m'),s(m'),k(m'))} < 1/2^{A(m',i(m'))} < 1/2^{S^3_{m',i(m')}} \\
&(\text{in particular } |u^*_{f(m',i(m'),s(m'),k(m')),j(m'),1}(x_0)| < 1/2^{S^3_{m',i(m')}}).
\end{aligned}$$

Using this fact, and also (3.17), (3.3) and (3.16), we have that

$$\begin{aligned}
&\Bigg\| w_{m'} - \sum_{h=1}^{2^{f(m',i(m'),s(m'),k(m'))}} y^*_{f(m',i(m'),s(m'),k(m')),j(m'),h}(x_0) \\
&\qquad \times y_{f(m',i(m'),s(m'),k(m')),j(m'),h} \Bigg\| \\
&\leq \Bigg\| \sum_{h=1}^{2^{f(m',i(m'),s(m'),k(m'))}} u^*_{f(m',i(m'),s(m'),k(m')),j(m'),h}(x_0) \\
&\qquad \times u_{f(m',i(m'),s(m'),k(m')),j(m'),h} \Bigg\| \\
&\quad + \|w_{m'} - k(m') x^*_{q_{m'}+i(m')}(x_0) w_{m',i(m'),s(m')}\| \\
&\quad + \|u^*_{f(m',i(m'),s(m'),k(m')),j(m'),1}(x_0) w_{m',i(m'),s(m')}\| \\
&\quad + \|k(m') x^*_{q_{m'}+i(m')}(x_0) u_{f(m',i(m'),s(m'),k(m^*)),j(m'),1}\| \\
&< 1/2^{S^3_{m',i(m')}} + \frac{1}{S_{m',i(m')}} + S_{m',i(m')}/2^{S^3_{m',i(m')}} + \frac{1}{2^{m+2}} < \frac{1}{2^{m+1}};
\end{aligned}$$

this, by (3.14), completes the proof of (3.15). Concluding case (A) we call again $\{m(k)\}$ our subsequence: Then the preceding procedure gives the existence, for each k, of a permutation

$$\text{(3.18)}\quad \begin{aligned}&\{\pi(n)\}_{n=q_{m(k)}+1}^{q_{m(k)+1}} \text{ of } \{n\}_{n=q_{m(k)}+1}^{q_{m(k)+1}}, \text{ an integer } q_{0,m(k)} \text{ with } q_{m(k)}+1\\ &\le q_{0,m(k)} \le q_{m(k)+1} \text{ and a positive number } \varepsilon'_{m(k)} \to 0 \text{ with } k,\\ &\text{such that } \Big\| x_0 - \Big\{ \sum_{n=1}^{q_{m(k)}} y_n^*(x_0)y_n + \sum_{n=q_{m(k)}+1}^{q_{0,m(k)}} y_{\pi(n)}^*(x_0)y_{\pi(n)} \Big\}\Big\| < \varepsilon'_{m(k)}\end{aligned}$$

(the proof of (3.15) indicates that the two facts which determine $\varepsilon'_{m(k)}$ are (3.14) and the first relation of (3.12)).

Case (B): Suppose that $\{m'(k)\}$ is infinite: For each $k|x^*_{q_{m'(k)}+j}(x_0)| \le 1/S^2_{m'(k),j}$ for $1 \le j \le r_{t(m'(k))+1} - r_{t(m'(k))}$, hence by (3.3) we have that

$$\Big\| \sum_{n=r_{t(m'(k))}+1}^{r_{t(m'(k))+1}} x_n^*(x_0)x_n \Big\| \le \sum_{n=r_{t(m'(k))}+1}^{r_{t(m'(k))+1}} \frac{1}{S^2_{m'(k),n-r_{t(m'(k))}}} < \frac{r_{t(m'(k))+1}}{S^2_{m'(k),1}} < \frac{1}{2^{m'(k)}}$$

and, by the second part of Reference I* of the Introduction, moreover by (3.1), setting $q_m = r_{t(m)}$ for each m and $m'(0) = q_0 = 0$, we have that

$$x_0 = \sum_{k=0}^{\infty} \sum_{n=q_{m'(k)}+1}^{q_{m'(k+1)}} x_n^*(x_0)x_n = \sum_{k=0}^{\infty} \sum_{n=q_{m'(k)}+1}^{q_{m'(k+1)}} y_n^*(x_0)y_n;$$

hence there exists, for each k, a positive number $\varepsilon'_{m'(k)}$ such that

$$\text{(3.19)}\qquad \Big\| x_0 - \sum_{n=1}^{q_{m'(k)}} y_n^*(x_0)y_n \Big\| < \varepsilon'_{m'(k)} \quad \text{with } \varepsilon'_{m'(k)} \to 0 \text{ with } k.$$

This completes the case B.

Our aim now is to show that, for each positive integer i,

$$\text{(3.20)}\quad \begin{aligned}&\text{setting } q(i) = q_{2i}, \text{ there exist a permutation } \{\pi(n)\}_{n=q(i)+1}^{q(i+1)} \text{ of}\\ &\{n\}_{n=q(i)+1}^{q(i+1)}, \text{ an integer } q(0,i) \text{ with } q(i)+1 \le q(0,i) \le q(i+1) \text{ and a}\\ &\quad \text{positive number } \varepsilon(i) \to 0 \text{ with } i, \text{ so that}\\ &\Big\| x_0 - \Big\{ \sum_{n=1}^{q(i)} y_n^*(x_0)y_n + \sum_{n=q(i)+1}^{q(0,i)} y_{\pi(n)}^*(x_0)y_{\pi(n)} \Big\}\Big\| < \varepsilon(i);\end{aligned}$$

hence, setting $q(0,0) = 0, x_0 = \sum_{i=0}^{\infty} \sum_{n=q(0,i)+1}^{q(0,i+1)} y^*_{\pi(n)}(x_0) y_{\pi(n)}$.

Suppose then that $\{m(k)\}$ and $\{m'(k)\}$ are both infinite; fix a positive integer i and consider $q(i) = q_{2i}$. We have two possibilities:

(a) $2i+1 = m(k)$ for some k. Then by (3.18) we have that

$$\Big\| x_0 - \Big\{ \sum_{n=1}^{q_{2i+1}} y^*_n(x_0) y_n + \sum_{n=q_{2i+1}+1}^{q_{0,m(k)}} y^*_{\pi(n)}(x_0) y_{\pi(n)} \Big\} \Big\| < \varepsilon'_{m(k)}$$

($q_{m(k)} = q_{2i+1}$) and (3.20) holds for $q(0,i) = q_{0,m(k)}$ and for $\{\pi(n)\}_{n=q(i)+1}^{q(0,i)} = \{n\}_{n=q_{2i}+1}^{q_{2i+1}} \cup \{\pi(n)\}_{n=q_{m(k)}+1}^{q(0,i)}$.

(b) $2i+1 = m'(k)$ for some k. In this case, by (3.19) we have that

$$\Big\| x_0 - \Big\{ \sum_{n=1}^{q(i)} y^*_n(x_0) y_n + \sum_{n=q(i)+1}^{q_{2i+1}} y^*_n(x_0) y_n \Big\} \Big\| < \varepsilon'_{m'(k)};$$

hence (3.20) holds for $q(0,i) = q_{m'(k)} = q_{2i+1}$ and for $\{\pi(n)\}_{n=q(i)+1}^{q(0,i)} = \{n\}_{n=q_{2i}+1}^{q_{2i+1}}$, while if one of the two sequences $\{m(k)\}$ or $\{m'(k)\}$ is finite (or does not appear), there exists a natural number i_0 such that for each $i \geq i_0$ we have either always (a) or always (b). Therefore setting, for each $i, \varepsilon(i) = \varepsilon'_{m(k)}$ if we are in (a) and $\varepsilon(i) = \varepsilon'_{m'(k)}$ if we are in (b), (3.20) is proved; hence, replacing i by m and $q(0,i)$ by $q'(m), \{y_n\}$ verifies the definition (D_4) of the Introduction; this completes the proof of Theorem 1.

4. Extension of the M-basis with controlled coefficients

We begin with a property of the Banach spaces of type > 1, which is analogous to the property of Lemma 2.2.

LEMMA 4.1: *Let X be a Banach space of type > 1. Then, for any subspace $Y, X/Y$ too has type > 1 and there exists an integer K such that the following two conditions hold:*

(a) *For each finite codimensional subspace U of X and for each positive integer $m, U = U_m + U_{0,m}$ with $U_m = \operatorname{span}\{e_n\}_{n=1}^m$, where $\{e_n + Y\}_{n=1}^m$ is $(1 + 1/2^m)$-equivalent to the natural basis of l_2^m. Moreover, there exists a projection $Q : U/Y \to U_m/Y$ with $\|Q\| < K$, such that $U_{0,m}/Y = (U/Y) \cap Q_\perp$.*

(b) *For each finite codimensional subspace Y_0 of Y there exists a finite codimensional subspace X_0 of X and an isomorphism $T : X_0/Y_0 \to X_0/Y$ with $\|T\| \cdot \|T^{-1}\| \leq \frac{19}{9}$; moreover, for each finite codimensional subspace*

U of X_0 and for each positive integer m, we have the same decomposition $U = U_m + U_{0,m}$ with the properties of (a), but where Y is replaced by Y_0 and K is replaced by $\frac{19}{9}K$ (in particular, $\frac{19}{9}K$ is independent of Y_0, but depends only on X and Y).

Proof: X/Y has type > 1 otherwise l_1 would be finitely represented in X/Y, hence in X too, by the same proof of (e) $\Rightarrow$ (d) of Proposition 1.1. Since (a) follows from Reference IV* of the Introduction, we go on to prove (b): Let $\{w_n\}_{n=1}^N, \{w_n^*\}_{n=1}^N$ and $\{F_n\}_{n=1}^N$ be sequences of X, X^* and $(X/Y_0)^*$ respectively, with $\{w_n + Y_0\}_{n=1}^N \frac{1}{10}$-dense in the unit sphere of Y/Y_0 and, for each n with $1 \leq n \leq N$, $\|F_n\| = 1 = \|w_n + Y_0\| = F_n(w_n + Y_0) = \|w_n^*\|$, where $w_n^*(x) = F_n(x + Y_0)$ for each x of X. Setting $X_0 = X \cap \{\bigcap_{n=1}^N w_{n\perp}^*\}$, we claim that there is an isomorphism $T : X_0/Y_0 \to X_0/Y$ with $\|T\| = 1$ and $\|T^{-1}\| \leq \frac{19}{9}$. Indeed, for each $x \in X_0$ and for each $y \in Y$, we have that $\|x + y + Y_0\| \geq \frac{9}{19}\|x + Y_0\|$ (hence $\|x + Y\| \geq \frac{9}{19}\|x + Y_0\|$) because, since $X_0/Y_0 = (X/Y_0) \cap \{\bigcap_{n=1}^N F_{n\perp}\}$,

$$\begin{aligned}\|x + y|Y_0\| \geq \tfrac{9}{10}\|y + Y_0\| &= \tfrac{9}{10}\| - \{(x + Y_0) - (x + y + Y_0)\}\| \\ &\geq \tfrac{9}{10}\|x + Y_0\| - \tfrac{9}{10}\|x + y + Y_0\|,\end{aligned}$$

that is $\frac{19}{10}\|x + y + Y_0\| \geq \frac{9}{10}]\|x + Y_0\|$. Therefore, for each $x \in X_0$, $\frac{9}{19}\|x + Y_0\| \leq \|x + Y\| \leq \|x + Y_0\|$. For the second part of (b), since U is also a finite codimensional subspace of X, if $U = U_m + U_{0,m}$ is the decomposition of (a), if $u(\in U) = u_m(\in U_m) + u_{0,m}(\in U_{0,m})$, we have that $\|u_m + Y_0\| \leq \frac{19}{9}\|u_m + Y\|$ (since $X_0 \supset U \supset U_m$) $\leq \frac{19}{9}K\|u_m + u_{0,m} + Y\|$ (by (a)) $= \frac{19}{9}K\|u + Y\| \leq \frac{19}{9}K\|u + Y_0\|$; this completes the proof of Lemma 4.1. ∎

Coming back to the general case, the next Proposition 4.1 establishes the extension of M-bases with controlled coefficients of Proposition 2.1. The next remark explains the reasons for the particular form of Proposition 4.1. The reader is advised to read the statement of the proposition before turning to the following.

Remark 4.1: In the proof of Proposition 4.1 two particular devices will appear. The first one is that, in order to define $\{v_n\}$, so that property (a) of Proposition 4.1 holds, we shall have to work, for each m, in $X/Y_{0,m}$ where $Y_{0,m}$ is a finite codimensional subspace of Y, while if X/Y has type > 1, passing from X/Y to $X/Y_{0,m}$ the constant K of condition (a) of Lemma 4.1 may change.

On the other hand, K will determine the nature of $\{v_n^*\}$, hence in order to get $\{w_n\}$ uniformly minimal, we need the same K for each $Y_{0,m}$. Therefore, instead of working in $X/Y_{0,m}$, we shall work in $X'_{0,m}/Y_{0,m}$ for a suitable finite codimensional subspace $X'_{0,m}$ of X; we shall obtain this $X'_{0,m}$ by means of the same procedure of the proof of condition (b) of Lemma 4.1, in order to have the same constant $\frac{19}{9}K$ for each $Y_{0,m}$.

The second device concerns property (b) of the assertion of Proposition 4.1, that is the definition of $\{u_n\}$; in this case, for each m, we shall work in the space $Y+Z_{m-1}$ where Z_{m-1} is a finite-dimensional subspace of X; the problem is that, if Y has type >1 (hence, by Lemma 2.1, if $Y+Z_{m-1}$ has type >1), a family C, of uniformly complemented subspaces of $Y+Z_{m-1}$, does not preserve in general this property when we consider these subspaces in X. Hence, again we have to consider, instead of X, a suitable finite codimensional subspace $X'_{0,m}$ of X, such that C preserves its properties also in $X'_{0,m}$.

These two devices are the reasons for the presence, for each m, of the sequence $\{v_{m,0,n}\}_{n=1}^{Q_{m,0}}$, since its aim is to get the desired finite codimensional subspace of X which ensures achieving the goals of both of these two devices. The reasoning for (c) will appear later in Lemma 4.2.

We shall use the following characterization (f) of [54] (p. 499), for an M-basis $\{x_n\}$ of X with $\{x_n, x_n^*\}$ biorthogonal:

$$\begin{aligned}(4.1)\qquad &\operatorname{span}\{x_n^*\} \text{ is } K\textit{-norming} \text{ on } X \text{ (that is,}\\ &\|x\| \le K\cdot\sup\{|x^*(x)| : x^*\in\operatorname{span}\{x_n^*\} \text{ and } \|x^*\|=1\} \text{ for each } x \text{ of } X)\\ &\Leftrightarrow \{x_n\} \text{ is } K\textit{-norming} \text{ (that is,}\\ &\|x\|\le K\cdot\sup\{\operatorname{dist}(x,\operatorname{span}\{x_n\}_{n\ge m}) : 1\le m<+\infty\} \text{ for each } x \text{ of } X).\end{aligned}$$

In particular it follows that, if $\{x_n\}$ is 1-norming, $\operatorname{dist}(x,\operatorname{span}\{x_n\}_{n\ge m})\to\|x\|$ with m for each x of X.

Proposition 4.1: *Let Y be a subspace of a separable Banach space X. Then there exist an integer $H\ge 4$ and a norming M-basis $\{w_n\}$ of X, with $\{w_n,w_n^*\}$ biorthogonal, $\|w_n\|=1$ and $\|w_n^*\|<H$ for each n, such that*

$$\begin{aligned}&\{w_n\}=\{u_n\}\cup\{v_n\} \text{ where } \{u_n\} \text{ is an } M\text{-basis of } Y;\\ &\{u_n,u_n^*\}=\{\{\{u_{m,j,n},u_{m,j,n}^*\}_{n=1}^{2^m}\}_{j=1}^{R_m}\cup\{u_{0,m,n},u_{0,m,n}^*\}_{n=1}^{P_m}\}_{m=1}^{\infty}\\ &\text{and } \{v_n,v_n^*\}=\{\{v_{m,0,n},v_{m,0,n}^*\}_{n=1}^{Q_{m,0}}\cup\{\{v_{m,j,n},v_{m,j,n}^*\}_{n=1}^{2^m}\}_{j=1}^{T_m}\\ &\qquad\cup\{v_{0,m,n},v_{0,m,n}^*\}_{n=1}^{Q_{0,m}}\}_{m=1}^{\infty}.\end{aligned}$$

Moreover, the following three properties hold:

(a) *For each $x_0(\ne 0)$ of X and for each m there exists an integer $j(0,m)$ with $1\le j(0,m)\le T_m$ such that $\sum_{n=1}^{2^m}|v_{m,j(0,m),n}^*(x_0)|<\|x_0\|/2^{m-1}$.*

(b) *For each* m *and for each* $x_0 (\neq 0)$ *of* X, *with* $\sum_{n=1}^{Q_{m,0}} |v^*_{m,0,n}(x_0)| < \|x_0\|/4$, *there exists an integer* $j(0,m)$ *with* $1 \leq j(0,m) \leq R_m$ *such that* $\sum_{n=1}^{2^m} |u^*_{m,j(0,m),n}(x_0)| < \|x_0\|/2^{m-1}$.

(c) *For each* m, *setting*

$$Y_{0,m} = Y \cap \{\bigcap_{k=1}^{m} \{\{\bigcap_{j=1}^{R_k} \bigcap_{n=1}^{2^k} u^*_{k,j,n\perp}\} \cap \{\bigcap_{n=1}^{P_k} u^*_{0,k,n\perp}\}\}\},$$

$\{\{\{u_{k,j,n} + Y_{0,m}\}_{n=1}^{2^k}\}_{j=1}^{R_k} \cup \{u_{0,k,n} + Y_{0,m}\}_{n=1}^{P_k}\}_{k=1}^{m} \cup \{v_n + Y_{0,m}\}$ *is a 1-norming* M*-basis of* $X/Y_{0,m}$.

Proof:

(4.2) Let $\{w'_n\} \cup \{w''_n\}$ be a norming M-basis of X, with $\{w'_n, w'^*_n\} \cup \{w''_n, w''^*_n\}$ biorthogonal, such that $\{w'_n\}$ is an M-basis of Y.

Assume that we have already defined, for a fixed integer $m > 1$,

(4.3)
$$B(m-1) = B'(m-1) \cup B''(m-1)$$

biorthogonal with

$$B'(m-1) = \{\{u_{k,n}, u^*_{k,n}\}_{n=1}^{R'_k} \cup \{u_{0,k,n}, u^*_{0,k,n}\}_{n=1}^{P_k}\}_{k=1}^{m-1}$$

and

$$B''(m-1) = \{\{v_{k,0,n}, v^*_{k,0,n}\}_{n=1}^{Q_{k,0}} \cup \{v_{k,n}, v^*_{k,n}\}_{n=1}^{T'_k} \cup \{v_{0,k,n}, v^*_{0,k,n}\}_{n=1}^{Q_{0,k}}\}_{k=1}^{m-1},$$

with

$$\bigcap_{k-1}^{m-1} \left\{\left\{\bigcap_{n=1}^{Q_{k,0}} v^*_{k,0,n\perp}\right\} \cap \left\{\bigcap_{n=1}^{T'_k} v^*_{k,n\perp}\right\} \cap \left\{\bigcap_{n=1}^{Q_{0,k}} v^*_{0,k,n\perp}\right\}\right\} \supset Y$$

and

$$Y \supset \{\{u_{k,n}\}_{n=1}^{R'_k} \cup \{u_{0,k,n}\}_{n=1}^{P_k}\}_{k=1}^{m-1};$$

let us fix k with $1 \leq k \leq m-1$, then: $\{u_{k,n}\}_{n=1}^{R'_k} = \{\{u_{k,j,n}\}_{n=1}^{2^k}\}_{j=1}^{R_k}$ and $\{v_{k,n}\}_{n=1}^{T'_k} = \{\{v_{k,j,n}\}_{n=1}^{2^k}\}_{j=1}^{T_k}$; $\|u_{k,n}\| = 1$ and $\|u^*_{k,n}\| < H$ for $1 \leq n \leq R'_k$, $\|u_{0,k,n}\| = 1$ and $\|u^*_{0,k,n}\| < 3$ for $1 \leq n \leq P_k$, $\|v_{k,0,n}\| = 1$ and $\|v^*_{k,0,n}\| < 4$ for $1 \leq n \leq Q_{k,0}$, $\|v_{k,n}\| = 1$ and $\|v^*_{k,n}\| < H$ for $1 \leq$

$n \leq T_k^*, \|v_{0,k,n}\| = 1$ and $\|v_{0,k,n}^*\| < 4$ for $1 \leq n \leq Q_{0,k}$; setting $U_k = \operatorname{span}\{u_{k,n}\}_{n=1}^{R_k'}$ and

$$U_{0,k} = \operatorname{span}\{u_{0,k,n}\}_{n=1}^{P_k},$$
$$V_{k,0} = \operatorname{span}\{v_{k,0,n}\}_{n=1}^{Q_{k,0}},$$
$$V_k = \operatorname{span}\{v_{k,n}\}_{n=1}^{T_k'}$$

and $V_{0,k} = \operatorname{span}\{v_{0,k,n}\}_{n=1}^{Q_{0,k}}, Y = Y_k + Y_{0,k}$ with $Y_k = \sum_{j=1}^k (U_j + U_{0,j})$ and

$$Y_{0,k} = Y \cap \Big\{ \bigcap_{j=1}^k \Big\{ \Big\{ \bigcap_{n=1}^{R_j'} u_{j,n,\perp}^* \Big\} \cap \Big\{ \bigcap_{n=1}^{P_j} u_{0,j,n\perp}^* \Big\} \Big\} \Big\},$$

$$Z_k = \sum_{j=1}^k (V_{j,0} + V_j + V_{0,j}), X = X_k + X_{0,k} \text{ with } X_k = Y_k + Z_k$$

and

$$X_{0,k} = X \cap \Big\{ \bigcap_{j=1}^k \Big\{ \Big\{ \bigcap_{n=1}^{R_j'} u_{j,n\perp}^* \Big\} \cap \Big\{ \bigcap_{n=1}^{P_j} u_{0,j,n\perp}^* \Big\}$$
$$\cap \Big\{ \bigcap_{n=1}^{Q_{j,0}} v_{j,0,n\perp}^* \Big\} \cap \Big\{ \bigcap_{n=1}^{T_j'} v_{j,n\perp}^* \Big\} \cap \Big\{ \bigcap_{n=1}^{Q_{0,j}} v_{0,j,n\perp}^* \Big\} \Big\} \Big\},$$

hence $Y_{0,k} = Y \cap X_{0,k}$; then $\operatorname{dist}(w_j', Y_k) < 1/2^k$ and $\operatorname{dist}(w_j'', X_k + Y) < 1/2^k$ for $1 \leq j \leq k$; setting moreover

$$B^*(k) = B'^*(k) \cup B''^*(k) \text{ with } B'^*(k) = \{\{u_{j,n}^*\}_{n=1}^{R_j'} \cup \{u_{0,j,n}^*\}_{n=1}^{P_j}\}_{j=1}^k$$

and

$$B''^*(k) = \{\{v_{j,0,n}^*\}_{n=1}^{Q_{j,0}} \cup \{v_{j,n}^*\}_{n=1}^{T_j'} \cup \{v_{0,j,n}^*\}_{n=1}^{Q_{0,j}}\}_{j=1}^k,$$

then

$$\operatorname{span}\{B^*(k)\} \supset \{w_j'^*\}_{j=1}^k \cup \{w_j''^*\}_{j=1}^k; \text{ the properties (a) and}$$

(b) are verified for k; finally there exists $\{w_{k,n}'''^*\}$ in X^*, with $w_{k,n\perp}'''^* \supset Y_{0,k}$ for each n, such that, setting $F_{k,n}'''(x + Y_{0,k}) = w_{k,n}'''^*(x)$ for each x of X and for each $n, \operatorname{span}\{F_{k,n}'''\}$ is 1-norming on $X/Y_{0,k}$, moreover such that $\operatorname{span}\{B'^*(k) \cup B''^*(m-1)\} \supset \{w_{k,n}'''^*\}_{n=1}^{m-1}$.

By (4.2) it is sufficient to prove the existence of

$$\{w'''^*_{m,n}\} \text{ in } X^* \text{ and of } \{u_{m,n}, u^*_{m,n}\}_{n=1}^{R'_m} \cup \{u_{0,m,n}, u^*_{0,m,n}\}_{n=1}^{P_m} \cup$$
$$\{v_{m,0,n}, v^*_{m,0,n}\}_{n=1}^{Q_{m,0}} \cup \{v_{m,n}, v^*_{m,n}\}_{n=1}^{T'_m} \cup \{v_{0,m,n}, v^*_{0,m,n}\}_{n=1}^{Q_{0,m}}$$

biorthogonal, such that (4.3) is verified also if $m-1$ is replaced by m (we point out that, for each m, if $\text{span}\{B'^*(m) \cup B''^*(\infty)\}(= \text{span}\{B'^*(m) \cup \{v_n^*\}\}) \supset \{w'''^*_{m,n}\}$, then $\{\{u_{j,n} + Y_{0,m}\}_{n=1}^{R'_j} \cup \{u_{0,j,n} + Y_{0,m}\}_{n=1}^{P_j}\}_{j=1}^{m} \cup \{v_n + Y_{0,m}\}$ is a 1-norming M-basis of $X/Y_{0,m}$): This again is the main part of a valid procedure by induction, since the first step (that is for $m = 1$) is only an obvious simplified version of the construction of the general step. The proof is organized in six parts, each of parts A, B, C, E and F dealing with definitions of the following parts of the system:
$\{u_{m,n}, u^*_{m,n}\}_{n=1}^{R'_m}$ (part A), $\{u_{0,m,n} u^*_{0,m,n}\}_{n=1}^{P_m}$ (part B),
$\{v_{m,0,n}, v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}$ (part C), $\{v_{m,n}, v^*_{m,n}\}_{n=1}^{T'_m}$ (part E),
$\{v_{0,m,n}, v^*_{0,m,n}\}_{n=1}^{Q_{0,m}}$ (part F); while part D verifies property (b).
Each part, in its turn, will be developed in several steps.

Part A:

To define $\{u_{m,n}, u^*_{m,n}\}_{n=1}^{R'_m}$ we follow step by step the procedure of the definition of $\{u_{m,n}, u^*_{m,n}\}_{n=1}^{S_m}$ of Proposition 2.1, that is we use Lemma 2.5, only here X and $\{y_n, y_n^*\}_{n=1}^{Q}$ of Lemma 2.5 have to be replaced by $Y + Z_{m-1}$ and by $B(m-1)$, respectively. In particular $(Y + Z_{m-1}) \cap (\text{span}\{B^*(m-1)\})_\perp = Y_{0,m-1}$, that is $Y + Z_{m-1} = X_{m-1} + Y_{0,m-1}$. Then, if Y has type > 1, we can use (2.8) where V has to be replaced by $Y_{0,m-1}$; hence H of (2.8) does not depend on m. However, if Y does not have type > 1, we can use Lemma 2.3. Therefore, by Remark 2.1,
(4.4)

$$\text{setting } U_m = \text{span}\{u_{m,n}\}_{n=1}^{R'_m} = \text{span}\{\{u_{m,j,n}\}_{n=1}^{2^m}\}_{j=1}^{R_m}$$

and

$$W_{0,m} = (Y + Z_{m-1}) \cap \left\{ \bigcap_{n=1}^{R'_m} u^*_{m,n\perp} \right\} = X_{m-1} + Y_{0,m-1} \cap \left\{ \bigcap_{n=1}^{R'_m} u^*_{m,n\perp} \right\},$$

hence $Y + Z_{m-1} = U_m + W_{0,m}$; then, for each sequence $\{\{a_{j,n}\}_{n=1}^{2^m}\}_{j=1}^{R_m}$ of numbers and for each positive number a, if there does not exist $j(0,m)$, with $1 \le j(0,m) \le R_m$, such that $\sum_{n=1}^{2^m} |a_{j(0,m),n}| < a/2^m$, it follows that $\text{dist}\left(\sum_{j=1}^{R_m} \sum_{n=1}^{2^m} a_{j,n} u_{m,j,n}, W_{0,m} \right) > a$

(where $W_{0,m}$ corresponds to U of Remark 2.1). We have to check that $\{u_{m,n}\}_{n=1}^{R'_m}$ verifies property (b), but this will be possible only later in part D, after the definition of $\{v_{m,0,n}\}_{n=1}^{Q_{m,0}}$.

Part B:

Proceeding as in the proof of Proposition 2.1, we proceed to define $\{u'_{0,m,n}\}_{n=1}^{P'_m}$ and $\{u'^*_{0,m,n}\}_{n=1}^{P'_m}$ such that $Y \supset \{u'_{0,m,n}\}_{n=1}^{P'_m}$ and $B(m-1) \cup \{u_{m,n}, u^*_{m,n}\}_{n=1}^{R'_m} \cup \{u'_{0,m,n}, u'^*_{0,m,n}\}_{n=1}^{P'_m}$ is biorthogonal, with $\|u'_{0,m,n}\| = 1$ for $1 \leq n \leq P'_m$ and $\text{dist}(w'_k, Y_{m-1} + \text{span}\{\{u_{m,n}\}_{n=1}^{R'_m} \cup \{u'_{0,m,n}\}_{n=1}^{P'_m}\}) < 1/2^m$ for $1 \leq k \leq m$. We have to consider w'^*_m and we point out that $(B^*(m-1))_{|Y} = (B'^*(m-1))_{|Y} \cup \{0\}$. Suppose that, setting

(4.5)
$$B'''^*(m) = B^*(m-1) \cup \{u^*_{m,n}\}_{n=1}^{R'_m} \cup \{u'^*_{0,m,n}\}_{n=1}^{P'_m}$$

and

$$B'''^*(m)_{|Y} = (B^*(m-1))_{|Y} \cup \{u^*_{m,n|Y}\}_{n=1}^{R'_m} \cup \{u'^*_{0,m,n|Y}\}_{n=1}^{P'_m},$$

then $w'^*_m \notin \text{span}\{B'''^*(m)\}$ but $w'^*_{m|_Y} \in \text{span}\{B'''^*(m)_{|Y}\}$, that is there exists $w'''^*_m \in \text{span}\{B'''^*(m)\}$ such that $w'^*_{m|Y} = w'''^*_{m|_Y}$,

hence $(w'^*_m - w'''^*_m)_\perp \supset Y$; then we set $P''_m = P'_m$ and we leave aside $w'^*_m - w'''^*_m$; we shall consider it later together with w''^*_m and we shall use the same procedure for both of them. While, if $w'^*_{m|Y} \notin \text{span}\{B'''^*(m)_{|Y}\}$, we set $P''_m = P'_m + 1$ and we define $u'_{0,m,P''_m} \in Y$ and $u'^*_{0,m,P''_m} \in X^*$ from w'^*_m, by means of the same procedure of Proposition 2.1 for the definition of v'_{m,P''_m} and v'^*_{m,P''_m}. Finally, if $w'^*_m \in \text{span}\{B'''^*(m)\}$, we again set $P''_m = P'_m$. We then pass, from $\{u'_{0,m,n}, u'^*_{0,m,n}\}_{n=1}^{P''_M}$ to $\{u_{0,m,n}, u^*_{0,m,n}\}_{n=1}^{P_m}$ with $Y \supset \{u_{0,m,n}\}_{n=1}^{P_m}$, by means of the same procedure we used in Proposition 2.1 to pass from $\{v'_{m,n}, v'^*_{m,n}\}_{n=1}^{P''_m}$ to $\{v_{m,n}, v^*_{m,n}\}_{n=1}^{P_m}$.

Part C:

We are now going to define $\{v_{m,0,n}, v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}$ and we shall do that in the following five steps.

(C_1) In the first step we describe both our starting point and our aim (we use (4.3)). Set

(4.6) $B'(m) = B'(m-1) \cup \{u_{m,n}, u^*_{m,n}\}_{n=1}^{R'_m} \cup \{u_{0,m,n}, u^*_{0,m,n}\}_{n=1}^{P_m}$ and $B'^*(m) = B'^*(m-1) \cup \{u^*_{m,n}\}_{n=1}^{R'_m} \cup \{u^*_{0,m,n}\}_{n=1}^{P_m}, X''_{0,m} = X_{0,m-1} \cap \left\{ \bigcap_{n=1}^{R'_m} u^*_{m,n\perp} \right\} \cap \left\{ \bigcap_{n=1}^{P_m} u^*_{0,m,n\perp} \right\}$ and $Y_{0,m} = Y_{0,m-1} \cap \left\{ \bigcap_{n=1}^{R'_m} u^*_{m,n\perp} \right\} \cap \left\{ \bigcap_{n=1}^{P_m} u^*_{0,m,n\perp} \right\} = Y_{0,m-1} \cap X''_{0,m}$; setting moreover $U_{0,m} = \text{span}\{u_{0,m,n}\}_{n=1}^{P_m}$ and $Y_m = Y_{m-1} +$

$U_m + U_{0,m}$, we have that $Y_m + Y_{0,m} = Y$ and $X = Z_{m-1} + Y_m + X''_{0,m} = U_m + X''_{0,m} + W_{0,m}$, since $W_{0,m} = X_{m-1} + U_{0,m} + Y_{0,m}$

(by (4.4)). Since we have to satisfy the two devices of Remark 4.1, our aim is to define a biorthogonal system $\{v_{m,0,n}, v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}$ with the following properties:

(4.7) $B'(m) \cup B''(m-1) \cup \{v_{m,0,n}, v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}$ is biorthogonal with $X \cap \left\{\bigcap_{n=1}^{Q_{m,0}} v^*_{m,0,n\perp}\right\} \supset Y$; $\|v_{m,0,m}\| = 1$ and $\|v^*_{m,0,n}\| < 4$ for $1 \leq n \leq Q_{m,0}$; setting moreover $X'_{0,m} = X''_{0,m} \cap \left\{\bigcap_{n=1}^{Q_{m,0}} v^*_{m,0,n\perp}\right\}$ and $V_{m,0} = \operatorname{span}\{v_{m,0,n}\}_{n=1}^{Q_{m,0}}$ (hence $X''_{0,m} = V_{m,0} + X'_{0,m}$ and, by (4.6), $X = U_m + V_{m,0} + X'_{0,m} + W_{0,m}$), for each $u \in U_m$ we have that $\operatorname{dist}(u, X'_{0,m} + W_{0,m}) \geq \frac{9}{10}\operatorname{dist}(u, W_{0,m})$; finally there is an isomorphism $T : X'_{0,m}/Y_{0,m} \to X'_{0,m}/Y$ with $\|T\| \cdot \|T^{-1}\| < \frac{19}{9}$.

The next steps occur in the space $X/Y_{0,m}$.

(C_2) The second step regards the second device of Remark 4.1; therefore, keeping our mind on (4.4), our aim is to obtain the middle property of (4.7). Going a moment to the space $X/W_{0,m}$ (where, by last relation of (4.6), $W_{0,m} \supset Y_{0,m}$), we find three sequences $\{w_{0,m,n}\}_{n=1}^{L_m}$ of U_m, $\{w^*_{0,m,n}\}_{n=1}^{L_m}$ of X^* and $\{F_{0,m,n}\}_{n=1}^{L_m}$ of $(X/W_{0,m})^*$, such that $\{w_{0,m,n} + W_{0,m}\}_{n=1}^{L_m}$ is $\frac{1}{10}$-dense in the unit sphere of $U_m/W_{0,m}$ and, for each n with $1 \leq n \leq L_m$,

$$\|w_{0,m,n} + W_{0,m}\| = F_{0,m,n}(w_{0,m,n} + W_{0,m}) = \|F_{0,m,n}\| = \|w^*_{0,m,n}\| = 1,$$

where $w^*_{0,m,n}(x) = F_{0,m,n}(x + W_{0,m})$ for each $x \in X$. Then

(4.8)
$$\operatorname{dist}\left(u, X''_{0,m} \cap \left\{\bigcap_{n=1}^{L_m} w^*_{0,m,n\perp}\right\} + W_{0,m}\right) \geq \tfrac{9}{10}\operatorname{dist}(u, W_{0,m}) \quad \text{for each } u \in U_m$$

(since

$$\begin{aligned}
&\operatorname{dist}(u, X''_{0,m} \cap \{\bigcap_{n=1}^{L_m} w^*_{0,m,n\perp}\} + W_{0,m}) \\
&= \operatorname{dist}(u + W_{0,m}, (X''_{0,m} \cap \{\bigcap_{n=1}^{L_m} w^*_{0,m,n\perp}\})/W_{0,m}) \\
&= \operatorname{dist}(u + W_{0,m}, (X''_{0,m}/W_{0,m}) \cap \{\bigcap_{n=1}^{L_m} F_{0,m,n\perp}\}) \\
&\geq \frac{9}{10}\|u + W_{0,m}\|).
\end{aligned}$$

So we have now defined the subspace $X''_{0,m} \cap \{\bigcap_{n=1}^{L_m} w^*_{0,m,n\perp}\}$ of $X''_{0,m}$, which satisfies the middle property of (4.7); but we need to obtain this subspace by means of a biorthogonal system $\{v'_{m,0,n}, v'^*_{m,0,n}\}_{n=1}^{Q'_{m,0}}$. Then our aim is to define $\{v'_{m,0,n}\}_{n=1}^{Q'_{m,0}}$ in $X''_{0,m}$ and $\{v'^*_{m,0,n}\}_{n=1}^{Q'_{m,0}}$ in X^* such that

$$(4.9)\quad \begin{aligned} &B'(m) \cup B''(m-1) \cup \{v'_{m,0,n}, v'^*_{m,0,n}\}_{n=1}^{Q'_{m,0}} \text{ is biorothogonal with} \\ &X \cap \Big\{ \bigcap_{n=1}^{Q'_{m,0}} v'^*_{m,0,n\perp} \Big\} \supset Y \text{ and } \|v'_{m,0,n} + Y_{0,m}\| = 1 \text{ for } 1 \le n \le Q'_{m,0}; \\ &\text{moreover such that } (X''_{0,m} \cap \{ \bigcap_{n=1}^{Q'_{m,0}} v'^*_{m,0,n\perp} \}) = \\ &\Big(X''_{0,m} \cap \Big\{ \bigcap_{n=1}^{L_m} w^*_{0,m,n\perp} \Big\} \Big); \text{ hence by (4.8), for each } u \in U_m, \\ &\operatorname{dist}\Big(u, X''_{0,m} \cap \Big\{ \bigcap_{n=1}^{Q'_{m,0}} v'^*_{m,0,n\perp} \Big\} + W_{0,m} \Big) \ge \tfrac{9}{10} \operatorname{dist}(u, W_{0,m}). \end{aligned}$$

Coming back to $X/Y_{0,m}$, we select three sequences $\{v'_{m,0,n}\}_{n=1}^{Q'_{m,0}}$ in $X''_{0,m}$, $\{v'^*_{m,0,n}\}_{n=1}^{Q'_{m,0}}$ in X^* and $\{F'_{m,0,n}\}_{n=1}^{Q'_{m,0}}$ in $(X/Y_{0,m})^*$, such that $\|v'_{m,0,n}+Y_{0,m}\| = 1$ and $v'^*_{m,0,n}(x) = F'_{m,0,n}(x + Y_{0,m})$ for each x of X and for each n with $1 \le n \le Q'_{m,0}$, moreover such that $\{v'_{m,0,n} + Y_{0,m}, F'_{m,0,n}\}_{n=1}^{Q'_{m,0}}$ is biorthogonal with $(X/Y_{0,m}) \cap \{\bigcap_{n=1}^{Q'_{m,0}} F'_{m,0,n\perp}\} \supset (Z_{m-1} + Y_m)/Y_{0,m}$ (hence with $X \cap \{\bigcap_{n=1}^{Q'_{m,0}} v'^*_{m,0,n\perp}\} \supset Z_{m-1} + Y_m + Y_{0,m} = Z_{m-1} + Y$, therefore the first part of (4.9) is verified); moreover such that

$$X''_{0,m}/Y_{0,m} = \operatorname{span}\{v'_{m,0,n} + Y_{0,m}\}_{n=1}^{Q'_{m,0}} + (X''_{0,m} \cap \{ \bigcap_{n=1}^{L_m} w^*_{0,m,n\perp} \})/Y_{0,m},$$

with $(X''_{0,m} \cap \{\bigcap_{n=1}^{Q'_{m,0}} v'^*_{m,0,n\perp}\})/Y_{0,m} (= (X''_{0,m}/Y_{0,m}) \cap \{\bigcap_{n=1}^{Q'_{m,0}} F'_{m,0,n\perp}\}) = (X''_{0,m} \cap \{\bigcap_{n=1}^{L_m} w^*_{0,m,n\perp}\})/Y_{0,m}$; this last fact implies that $X''_{0,m} \cap \{\bigcap_{n=1}^{Q'_{m,0}} v'^*_{m,0,n\perp}\} = X''_{0,m} \cap \{\bigcap_{n=1}^{L_m} w^*_{0,m,n\perp}\}$ since $X''_{0,m} \supset Y_{0,m}, \{\bigcap_{n=1}^{Q'_{m,0}} v'^*_{m,0,n\perp}\} \supset Y_{0,m}$ and $\{\bigcap_{n=1}^{L_m} w^*_{0,m,n\perp}\} \supset W_{0,m} \supset Y_{0,m}$; this completes the proof of (4.9).

(C_3) The third step regards the first device of Remark 4.1. Precisely, our aim is to obtain the last property of (4.7): If there already exists an isomorphism $T : (X''_{0,m} \cap \{\bigcap_{n=1}^{Q'_{m,0}} v'^*_{m,0,n\perp}\})/Y_{0,m} \to (X''_{0,m} \cap \{\bigcap_{n=1}^{Q'_{m,0}} v'^*_{m,0,n\perp}\})/Y$ with $\|T\| \cdot \|T^{-1}\| < \frac{19}{9}$, we set $Q''_{m,0} = Q'_{m,0}$ and $X'''_{0,m} = X''_{0,m} \cap \{\bigcap_{n=1}^{Q''_{m,0}} v'^*_{m,0,n\perp}\}$. Otherwise, following the proof of (b) of Lemma 4.1 (where now Y_0 and X are replaced by $Y_{0,m}$ and by $X''_{0,m} \cap \{\bigcap_{n=1}^{Q'_{m,0}} v'^{*}_{m,0,n\perp}\}$ respectively) and by means of the procedure of the first step, we obtain two further sequences $\{v'_{m,0,n}\}_{n=Q'_{m,0}+1}^{Q''_{m,0}}$ in X and $\{v'^{*}_{m,0,n}\}_{n=Q'_{m,0}+1}^{Q''_{m,0}}$ in X^* such that

$$(4.10)\quad \begin{aligned} &B'(m) \cup B''(m-1) \cup \{v'_{m,0,n}, v'^{*}_{m,0,n}\}_{n=1}^{Q''_{m,0}} \text{ is again biorthogonal} \\ &\text{with } \|v'_{m,0,n} + Y_{0,m}\| = 1 \text{ for } Q'_{m,0}+1 \le n \le Q''_{m,0} \text{ and} \\ &X \cap \Big\{ \bigcap_{n=Q'_{m,0}+1}^{Q''_{m,0}} v'^{*}_{m,0,n\perp} \Big\} \supset Y; \text{ moreover such that, setting} \\ &X'''_{0,m} = X''_{0,m} \cap \Big\{ \bigcap_{n=1}^{Q''_{m,0}} v'^{*}_{m,0,n\perp} \Big\}, \text{ there is an isomorphism} \\ &T : (X'''_{0,m}/Y_{0,m}) \to (X'''_{0,m}/Y) \text{ with } \|T\| \cdot \|T^{-1}\| < \tfrac{19}{9}. \end{aligned}$$

The aim of the last two steps is to obtain the properties of (4.7), regarding $\|v_{m,0,n}\|$ and $\|v^*_{m,0,n}\|$ for $1 \le n \le Q_{m,0}$.

(C_4) In the fourth step we continue to work in the space $X/Y_{0,m}$ and, by means of the preceding procedure and by Reference III* of the Introduction, there exist $\{v'_{m,0,n}\}_{n=Q''_{m,0}+1}^{Q_{m,0}}$ in X and $\{v'^{*}_{m,0,n}\}_{n=Q''_{m,0+1}}^{Q_{m,0}}$ in X^* such that $B'(m) \cup B''(m-1) \cup \{v'_{m,0,n}, v'^{*}_{m,0,n}\}_{n=1}^{Q_{m,0}}$ is biorthogonal again with $X \cap \{\bigcap_{n=Q''_{m,0}+1}^{Q_{m,0}} v'^{*}_{m,0,n\perp}\} \supset Y$; moreover now $\|v'_{m,0,n} + Y_{0,m}\| = \|v'^{*}_{m,0,n}\| = 1$ for $Q''_{m,0}+1 \le n \le Q_{m,0}$ and $Q_{m,0} \ge Q''_{m,0}(1 + 4^{\max\{\|v'^{*}_{m,0,n}\|: 1 \le n \le Q''_{m,0}\}})$. Then, by means of Reference II* of the Introduction, we pass to $\{v''_{m,0,n}, v''^{*}_{m,0,n}\}_{n=1}^{Q_{m,0}}$ with

$$(4.11)\quad \begin{aligned} &\operatorname{span}\{v''_{m,0,n} + Y_{0,m}\}_{n=1}^{Q_{m,0}} = \operatorname{span}\{v'_{m,0,n} + Y_{0,m}\}_{n=1}^{Q_{m,0}}, \\ &\operatorname{span}\{v''^{*}_{m,0,n}\}_{n=1}^{Q_{m,0}} = \operatorname{span}\{v'^{*}_{m,0,n}\}_{n=1}^{Q_{m,0}}, \\ &\|v''_{m,0,n} + Y_{0,m}\| = 1 \quad \text{and} \quad \|v''^{*}_{m,0,n}\| < 3 \quad \text{for } 1 \le n \le Q_{m,0} \end{aligned}$$

(hence $B'(m) \cup B''(m-1) \cup \{v''_{m,0,n}, v''^{*}_{m,0,n}\}_{n=1}^{Q_{m,0}}$ continues to be biorthogonal with $X \cap \{\bigcap_{n=1}^{Q_{m,0}} v''^{*}_{m,0,n\perp}\} \supset Y$).

(C_5) In the last step we pass from $X/Y_{0,m}$ to X and it is sufficient to choose, for each n with $1 \le n \le Q_{m,0}, v'''_{m,0,n}$ in $Y_{0,m}$ such that $\|v''_{m,0,n} + v'''_{m,0,n}\| < 4/3$; setting then $v_{m,0,n} = (v''_{m,0,n} + v'''_{m,0,n})/\|v''_{m,0,n} + v'''_{m,0,n}\|$ and $v^*_{m,0,n} = v''^{*}_{m,0,n}\|v''_{m,0,n} + v'''_{m,0,n}\|$, since $(\operatorname{span}\{B'^*(m) \cap B''^*(m-1)\})_\perp \cap \{\bigcap_{n=1}^{Q_{m,0}} v''^{*}_{m,0,n\perp}\} \supset Y_{0,m} \supset \{v'''_{m,0,n}\}_{n=1}^{Q_{m,0}}$, by (4.9), (4.10) and (4.11) we obtain that all the properties of (4.7) are verified.

Part D:

We are now ready to verify the property (b) for $\{u_{m,n}\}_{n=1}^{R'_m}$: For each $x_0, (\neq 0)$ of X, such that the hypothesis of (b) is verified, since by (4.7) $X = U_m + V_{m,0} + X'_{0,m} + W_{0,m}$, by the last but one property of (4.7) we have that

$$\begin{aligned}
\|x_0\| &\geq \operatorname{dist}\Big(\sum_{j=1}^{R'_m} u^*_{m,n}(x_0)u_{m,n} + \sum_{n=1}^{Q_{m,0}} v^*_{m,0,n}(x_0)v_{m,0,n}, X'_{0,m} + W_{0,m}\Big)\\
&\geq \operatorname{dist}\Big(\sum_{j=1}^{R'_m} u^*_{m,n}(x_0)u_{m,n}, X'_{0,m} + W_{0,m}\Big) - \Big\|\sum_{n=1}^{Q_{m,0}} v^*_{m,0,n}(x_0)v_{m,0,n}\Big\|\\
&\geq \operatorname{dist}\Big(\sum_{j=1}^{R'_m} u^*_{m,n}(x_0)u_{m,n}, X'_{0,m} + W_{0,m}\Big) - \sum_{n=1}^{Q_{m,0}} |v^*_{m,0,n}(x_0)|\\
&> \operatorname{dist}\Big(\sum_{j=1}^{R'_m} u^*_{m,n}(x_0)u_{m,n}, X'_{0,m} + W_{0,m}\Big) - \|x_0\|/4,
\end{aligned}$$

that is

$$\begin{aligned}
\|x_0\| &> \frac{4}{5}\operatorname{dist}\Big(\sum_{j=1}^{R'_m} u^*_{m,n}(x_0)u_{m,n}, X'_{0,m} + W_{0,m}\Big)\\
&> \frac{4}{5}\frac{9}{10}\operatorname{dist}\Big(\sum_{j=1}^{R'_m} u^*_{m,n}(x_0)u_{m,n}, W_{0,m}\Big)\\
&> \frac{1}{2}\operatorname{dist}\Big(\sum_{j=1}^{R'_m} u^*_{m,n}(x_0)u_{m,n}, W_{0,m}\Big).
\end{aligned}$$

We claim that there exists $j(0,m)$ such that $\sum_{n=1}^{2^m} |u^*_{m,j(0,m),n}(x_0)| < \|x_0\|/2^{m-1} = 2\|x_0\|/2^m$; indeed, if this were not so, by what we have just found, by (4.4) for $a = 2\|x_0\|$ and for $\{\{a_{j,n}\}_{n=1}^{2^m}\}_{j=1}^{R_m} = \{\{u^*_{m,j,n}(x_0)\}_{n=1}^{2^m}\}_{j=1}^{R_m}$, it would follow that

$$\|x_0\| > \frac{1}{2}\operatorname{dist}\Big(\sum_{j=1}^{R'_m} u^*_{m,n}(x_0)u_{m,n}, W_{0,m}\Big) > \frac{1}{2}2\|x_0\| = \|x_0\|.$$

Part E:

Going on to define $\{v_{m,n}, v^*_{m,n}\}_{n=1}^{T'_m}$, we are in a situation analogous to the situation of the definition of $\{u_{m,n}, u^*_{m,n}\}_{n=1}^{R'_m}$. Hence we follow again the procedure of the proof of Proposition 2.1, that is we use Lemma 2.5; only now

X, span$\{y_n\}_{n=1}^{Q}$ and $X \cap \{\bigcap_{n=1}^{Q} y_{n\perp}^*\}$ of Lemma 2.5 are replaced by $X/Y_{0,m}$, $(Z_{m-1}+Y_m+V_{m,0})/Y_{0,m}$ and $X'_{0,m}/Y_{0,m}$, respectively.

In particular, if X/Y has type > 1, by the last part of (4.7), we can use (b) of Lemma 4.1 where (in order to use a formula analogous to (2.8), see also the statement of Lemma 2.4) we can replace $\frac{19}{9}K$ by means of $(\frac{3}{4}H)/8$ and we can suppose $\frac{3}{4}H \geq 3$; in particular this number H does not depend on m. Then we proceed in two steps:

(E_1) In the first step we are in $X/Y_{0,m}$ and we can define $\{v'_{m,n}\}_{n=1}^{T'_m}$ in $X'_{0,m}$, $\{v'^*_{m,n}\}_{n=1}^{T'_m}$ in X^* and $\{F_{m,n}\}_{n=1}^{T'_m}$ in $(X/Y_{0,m})^*$, with the following properties: $\{v'_{m,n}+Y_{0,m}, F_{m,n}\}_{n=1}^{T'_m}$ is biorthogonal with $\{v'_{m,n}\}_{n=1}^{T'_m} = \{\{v'_{m,j,n}\}_{n=1}^{2^m}\}_{j=1}^{T_m}$ and, for each n with $1 \leq n \leq T'_m$, $v'^*_{m,n}(x) = F_{m,n}(x+Y_{0,m})$ for each x of X, $\|v'_{m,n}+Y_{0,m}\| = 1$ and

$$\|v'^*_{m,n}\| = \|F_{m,n}\| < \frac{3}{4}H; (X/Y_{0,m}) \cap \left\{ \bigcap_{n=1}^{T'_m} F_{m,n\perp} \right\} \supset \{Z_{m-1}+Y_m+V_{m,0}\}/Y_{0,m}$$

(hence $X \cap \{\bigcap_{n=1}^{T'_m} v'^*_{m,n\perp}\} \supset Z_{m-1}+V_{m,0}+Y$); for each $x_0 (\neq 0)$ of X, there exists $j(0,m)$ with $1 \leq j(0,m) \leq T_m$ such that $\sum_{n=1}^{2^m} |F_{m,j}(m,0,n)(x_0+Y_{0,m})|$ ($< \|x_0+Y_{0,m}\|/2^m$ if $x_0 \notin Y_{0,m}$ and otherwise $= 0$) $< \|x_0\|/2^m$.

(E_2) In the second step we pass to the space X and it is sufficient to choose, for each n with $1 \leq n \leq T'_{m'}$ (see the preceding procedure for $\{v''_{m,0,k}\}_{k=1}^{Q_{m,0}}$}), $v'_{m,n}$ in $v'_{m,n}+Y_{0,m}$ such that $\|v'_{m,n}\| < (4/3)\|v'_{m,n}+Y_{0,m}\| = 4/3$; then we set $v_{m,n} = v'_{m,n}/\|v'_{m,n}\|$ and $v^*_{m,n} = v'^*_{m,n}\|v'_{m,n}\|$; therefore, by what we have specified above, $\|v_{m,n}\| = 1$ and $\|v^*_{m,n}\| < H$ (with $H \geq 4$) for $1 \leq n \leq T'_m$; $B'(m) \cup B''(m-1) \cup \{v_{m,0,n}, v^*_{m,0,n}\}_{n=1}^{Q_{m,0}} \cup \{v_{m,n}, v^*_{m,n}\}_{n=1}^{T'_m}$ is biorthogonal with $X \cap \{\bigcap_{n=1}^{T'_m} v^*_{m,n\perp}\} \supset Y$; moreover (a) is verified since, for each $x_0 (\neq 0)$ of X, there exists $j(0,m)$ with $1 \leq j(0,m) \leq T_m$ such that

$$\begin{aligned}\sum_{n=1}^{2^m} |v^*_{m,j(m,0),n}(x_0)| &= \sum_{n=1}^{2^m} \|v'_{m,j(m,0),n}\| \cdot |v'^*_{m,j(m,0),n}(x_0)| \\ &< \frac{4}{3}\sum_{n=1}^{2^m} |v'^*_{m,j(m,0),n}(x_0)| = \frac{4}{3}\sum_{n=1}^{2^m} |F_{m,j(m,0),n}(x_0+Y_{0,m})| < \frac{\|x_0\|}{2^{m-1}}.\end{aligned}$$

Part F:

Finally we go on to define $\{v_{0,m,n}, v^*_{0,m,n}\}_{n=1}^{Q_{0,m}}$ and $\{w'''^*_{m,n}\}$, again proceeding in four steps:

(F_1) The first step is in $X/Y_{0,m}$ and, setting span$\{v_{m,n}\}_{n=1}^{T'_m} = V_m$, the procedure indicated above gives $\{v'_{0,m,n}\}_{n=1}^{Q'_{0,m}}$ in X and $\{v'^*_{0,m,n}\}_{n=1}^{Q'_{0,m}}$ in X^* with

$\|v'_{0,m,n}+Y_{0,m}\| = 1$ for $1 \le n \le Q'_{0,m}$, $B'(m)\cup B''(m-1)\cup\{v_{m,0,n}, v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}\cup \{v_{m,n}, v^*_{m,n}\}_{n=1}^{T'_m}\cup\{v'_{0,m,n}, v'^*_{0,m,n}\}_{n=1}^{Q'_{0,m}}$ biorthogonal, $X\cap\{\bigcap_{n=1}^{Q'_{0,m}} v'^*_{0,m,n\perp}\}\supset Y$ and

$$\operatorname{dist}(w''_k+Y_{0,m},(Z_{m-1}+Y_m+V_{m,0}+V_m)/Y_{0,m}+\operatorname{span}\{v'_{0,m,n}+Y_{0,m}\}_{n=1}^{Q'_{0,m}}) =$$
$$\operatorname{dist}(w''_k, Z_{m-1}+V_{m,0}+V_m+\operatorname{span}\{v'_{0,m,n}\}_{n=1}^{Q'_{0,m}}+Y) < 1/2^m \quad\text{for } 1\le k\le m.$$

(F_2) The second step is in X and let $\{F'''_{m,n}\}$ be a sequence of $(X/Y_{0,m})^*$ such that $\operatorname{span}\{F'''_{m,n}\}$ is 1-norming on $X/Y_{0,m}$; setting $w'''^*_{m,n}(x) = F'''_{m,n}(x+Y_{0,m})$ for each x of X and for each n, we have that $w'''^*_{m,n\perp}\supset Y_{0,m}$ for each n. Working in X, the same procedure of Proposition 2.1, for the definition of $\{v'_{m,P''_m}, v'^*_{m,P''_m}\}$, defines $\{v'_{0,m,n}\}_{n=Q'_{0,m}+1}^{Q''_{0,m}}$ in X and $\{v'^*_{0,m,n}\}_{n=Q'_{0,m}+1}^{Q''_{0,m}}$ in X^*, with $Q'_{0,m}\le Q''_{0,m}\le Q'_{0,m}+2m+1$, such that $B'(m)\cup B''(m-1)\cup\{v_{m,0,n}, v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}\cup\{v_{m,n}, v^*_{m,n}\}_{n=1}^{T'_m}\cup\{v'_{0,m,n}, v'^*_{0,m,n}\}_{n=1}^{Q''_{0,m}}$ is biorthogonal, with $X\cap\{\bigcap_{n=Q'_{0,m}+1}^{Q''_{0,m}} v'^*_{0,m,n\perp}\}\supset Y$ and with $\|v'_{0,m,n}+Y_{0,m}\| = 1$ for $Q'_{0,m}+1\le n\le Q''_{0,m}$; moreover so that $\operatorname{span}\{\{B'^*(m)\cup B''^*(m-1)\cup\{v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}\cup\{v^*_{m,n}\}_{n=1}^{T'_m}\cup\{v'^*_{0,m,n}\}_{n=1}^{Q''_{0,m}}\}\}\supset\{w''^*_m(\cup(w'^*_m-w'''^*_m)$ if in the case of (4.5)$)\}\cup\{\{w'''^*_{j,n}\}_{n=1}^m\}_{j=1}^m$.

In order to verify the last relation of (4.3), for m instead of $m-1$, it is sufficient to check that, for each fixed k with $1\le k\le m-1$ and for each n with $1\le n\le m$, $w'''^*_{k,n}\in\operatorname{span}\{\{B'^*(k)\cup B''^*(m-1)\cup\{v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}\cup\{v^*_{m,n}\}_{n=1}^{T'_m}\cup\{v'^*_{0,m,n}\}_{n=1}^{Q''_{0,m}}\}\}$ (since till now we only know that this formula holds if $B'^*(k)$ is replaced by $B'^*(m)$). For this, setting $w'''^*_{k,n} = w'''^*_{1,k,n}+w'''^*_{2,k,n}$ with $w'''^*_{1,k,n}\in\operatorname{span}\{\{B'^*(k)\cup B''^*(m-1)\cup\{v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}\cup\{v^*_{m,n}\}_{n=1}^{T'_m}\cup\{v'^*_{0,m,n}\}_{n=1}^{Q''_{0,m}}\}\}$ and with $w'''^*_{2,k,n}\in\operatorname{span}\{\{u^*_{j,n}\}_{n=1}^{R'_j}\cup\{u^*_{0,j,n}\}_{n=1}^{P_j}\}_{j=k+1}^m$, it is sufficient to check that $w'''^*_{2,k,n}=0$: Indeed $w'''^*_{k,n\perp}\supset Y_{0,k}$ by the definition of $\{w'''^*_{k,n}\}$ and $w'''^*_{1,k,n\perp}\supset Y_{0,k}$ too by the definition of $w'''^*_{1,k,n}$, hence we have that $w'''^*_{2,k,n\perp}\supset Y_{0,k}\supset\{\{u_{j,n}\}_{n=1}^{R'_j}\cup\{u_{0,j,n}\}_{n=1}^{P_j}\}_{j=k+1}^m$, which implies that $w'''^*_{2,k,n}=0$.

(F_3) In the third step we come back to the space $X/Y_{0,m}$ and, by References III* and II* of the Introduction, we can pass, from $\{v'_{0,m,n}, v'^*_{0,m,n}\}_{n=1}^{Q''_{0,m}}$, at first to $\{v'_{0,m,n}, v'^*_{0,m,n}\}_{n=1}^{Q_{0,m}}$ with $B'(m)\cup B''(m-1)\cup\{v_{m,0,n}, v^*_{m,0,n}\}_{n=1}^{Q_{m,0}}\cup\{v_{m,n}, v^*_{m,n}\}_{n=1}^{T'_m}\cup\{v'_{0,m,n}, v'^*_{0,m,n}\}_{n=1}^{Q_{0,m}}$ biorthogonal and with $X\cap\{\bigcap_{n=Q''_{0,m}+1}^{Q_{0,m}} v'^*_{0,m,n\perp}\}\supset Y$, such that $\|v'_{0,m,n}+Y_{0,m}\| = 1 = \|v'^*_{0,m,n}\|$ for $Q''_{0,m}+1\le n\le Q_{0,m}$ and $Q_{0,m}-Q''_{0,m}\ge Q''_{0,m}4^{\max\{\|v'^*_{0,m,n}\|:1\le n\le Q''_{0,m}\}}$; then we pass to $\{v''_{0,m,n}, v''^*_{0,m,n}\}_{n=1}^{Q_{0,m}}$ with $\operatorname{span}\{v''_{0,m,n}+Y_{0,m}\}_{n=1}^{Q_{0,m}} = \operatorname{span}\{v'_{0,m,n}+$

$Y_{0,m}\}_{n=1}^{Q_{0,m}}$ and $\operatorname{span}\{v''^*_{0,m,n}\}_{n=1}^{Q_{0,m}} = \operatorname{span}\{v'^*_{0,m,n}\}_{n=1}^{Q_{0,m}}$, such that, for each n with $1 \le n \le Q_{0,n}$, $\|v''_{0,m,n} + Y_{0,m}\| = 1$ and $\|v''^*_{0,m,n}\| < 3$.

(F_4) In the last step we pass to the space X choosing $v''_{0,m,n}$ in $v''_{0,m,n} + Y_{0,m}$ such that $\|v''_{0,m,n}\| < 4/3$ and setting $v_{0,m,n} = v''_{0,m,n}/\|v''_{0,m,n}\|$ and $v^*_{0,m,n} = v''^*_{0,m,n}\|v''_{0,m,n}\|$ for $1 \le n \le Q_{0,m}$. This completes the proof of Proposition 4.1. ∎

The next lemma concerns a property of the norming M-bases of the preceding kind and it provides the third and last device necessary for the proof of Theorem II. Moreover, the reason for part (c) of Proposition 4.1 becomes clear. Practically, this lemma improves the properties of the sequence $\{r_m\}$ of Reference I* of the Introduction.

LEMMA 4.2: *Let $\{x_n\} = \{x'_n\} \cup \{x''_n\}$ be an M-basis of X, with $\{x_n, x^*_n\} = \{x'_n, x'^*_n\} \cup \{x''_n, x''^*_n\}$ biorthogonal, $\|x_n\| = 1$ and $\|x^*_n\| < H$ for each n; moreover $\{x'_n\}$ is an M-basis of a subspace Y of X and there exists a subsequence $\{r'_m\}$ of $\{m\}$ such that, for each m, setting $Y_{0,m} = Y \cap \{\bigcap_{n=1}^{r'_m} x'^*_{n\perp}\}$, $\{x'_n + Y_{0,m}\}_{n=1}^{r'_m} \cup \{x''_n + Y_{0,m}\}$ is a 1-norming M-basis of $X/Y_{0,m}$.*

Then there exists a subsequence $\{r_m\}$ of $r'_m\}$ such that: If x_0 is an element of X such that there exists an increasing sequence $\{t(m)\}$ of positive integers, with $\sum_{n=r_{t(m)}+1}^{r_{t(m)+1}} |x''^*_n(x_0)| \to 0$ with m, it follows that

$$\operatorname{dist}\,(x_0 - \sum_{n=1}^{r_{t(m)}} [x'^*_n(x_0)x'_n + x''^*_n(x_0)x''_n], \operatorname{span}\{x'_n\}_{n=r_{t(m)}+1}^{r_{t(m)+1}}) \to 0 \quad \textit{with } m$$

(that is, for each m, the elements x''_n for $r_{t(m)}+1 \le n \le r_{t(m)+1}$ are not necessary for this approximation).

Proof: We begin by claiming that there exists an increasing sequence $\{p(m)\}$ of positive integers such that,

$$\begin{aligned} &\text{for each } m \text{ and for each } x \text{ of span } \{x'_n, x''_n\}_{n=1}^{r'_m}, \\ (4.12)\qquad &|\operatorname{dist}(x, \operatorname{span}\{\{x'_n\}_{n>r'_m} \cup \{x''_n\}_{n>r'_m+p(m)}\}) \\ &- \operatorname{dist}(x, \operatorname{span}\{x'_n\}_{n=r'_m+1}^{r'_{m+p(m)}})| \le \|x\|/\{2^{m+1}(1 + r'_m H)\}. \end{aligned}$$

Indeed fix m, x of the unit sphere of $\operatorname{span}\{x'_n, x''_n\}_{n=1}^{r'_m}$ and set

$$a = 1/\{2^{m+1}(1 + r'_m H)\};$$

since $\{x'_n+Y_{0,m}\}_{n=1}^{r'_m} \cup \{x''_n+Y_{0,m}\}$ is 1-norming, by (4.1) there is an integer $p(m) = p(m,x)$ such that $|\operatorname{dist}(x+Y_{0,m}, \operatorname{span}\{x''_n+Y_{0,m}\}_{n>r'_m+p(m)}) - \|x+Y_{0,m}\|| < a/2$, that is $|\operatorname{dist}(x, \operatorname{span}\{x''_n\}_{n>r'_m+p(m)} + Y_{0,m}) - \operatorname{dist}(x, Y_{0,m})|) < a/2$; moreover, we can also choose $p(m)$ such that

$$|\operatorname{dist}(x, Y_{0,m}) - \operatorname{dist}(x, \operatorname{span}\{x'_n\}_{n=r'_m+1}^{r'_m+p(m)})| < a/2;$$

hence we conclude that

$$\begin{aligned}
&\operatorname{dist}(x, \operatorname{span}\{\{x'_n\}_{n>r'_m} \cup \{x''_n\}_{n>r'_m+p(m)}\}) \\
&\leq \operatorname{dist}(x, \operatorname{span}\{x'_n\}_{n=r'_m+1}^{r'_m+p(m)}) \\
&< \operatorname{dist}(x, Y_{0,m}) + a/2 \\
&< \operatorname{dist}(x, \operatorname{span}\{x''_n\}_{n>r'_m+p(m)} + Y_{0,m}) + a \\
&= \operatorname{dist}(x, \operatorname{span}\{\{x'_n\}_{n>r'_m} \cup \{x''_n\}_{n>r'_m+p(m)}\}) + a;
\end{aligned}$$

since $\operatorname{span}\{x'_n, x''_n\}_{n=1}^{r'_m}$ has finite dimension, it is possible to choose $p(m)$ independent of x, which completes the proof of (4.12). By (4.12), setting $r_1 = r'_1, r_2 = r'_{1+p(1)}, r_3 = r'_{(1+p(1))+p(1+p(1))}$ and so on, we have that

$$\begin{aligned}
&\text{for each } m \text{ and for each } x \text{ of span } \{x'_n, x''_n\}_{n=1}^{r_m}, \\
&|\operatorname{dist}(x, \operatorname{span}\{\{x'_n\}_{n>r_m} \cup \{x''_n\}_{n>r_{m+1}}\}) \\
&\quad - \operatorname{dist}(x, \operatorname{span}\{x'_n\}_{n=r_m+1}^{r_{m+1}})| \leq \|x\|/\{2^{m+1}(1 + r_m H)\}.
\end{aligned} \tag{4.13}$$

If $x_0 \in X$, such that the hypothesis of the lemma is verified, we can assume that $\sum_{n=r_{t(m)}+1}^{r_{t(m)+1}} |x''^*_n(x_0)| < 1/2^m$ for each m; now it is sufficient to prove that there exists another increasing sequence $\{s(m)\}$ of positive integers such that, for each m,

$$\left\| x_0 - \left\{ \sum_{n=1}^{r_{t(s(m))}} [x'^*_n(x_0)x'_n + x''^*_n(x_0)x''_n] + v_m \right\} \right\| < 1/2^m$$

with $v_m \in \operatorname{span}\{x'_n\}_{n=r_{t(s(m))}+1}^{r_{t(s(m))+1}}$. Then we set

$$\begin{aligned}
&x_0 = x' + x'' \text{ with } x'' = \sum_{m=1}^{\infty} \sum_{n=r_{t(m)}+1}^{r_{t(m)+1}} x''^*_n(x_0)x''_n, \\
&\text{hence } x''^*_n(x') = 0 \text{ for } r_{t(m)} + 1 \leq n \leq r_{t(m)+1} \text{ for each } m.
\end{aligned} \tag{4.14}$$

There exists then a subsequence $\{s(m)\}$ of $\{m\}$ such that, for each m,

(i) there exists $x'_{0,m}$ in $\operatorname{span}\{x'_n, x''_n\}_{n=1}^{r_{t(s(m))}}$ with $\|x' - x'_{0,m}\| < 1/2^{m+3}$;

(ii) $2^{s(m)} > \max\{1, \|x'\|\}2^{m+4}$;

(iii) $\|x'' - A_2\| < 1/2^{m+1}$ with $A_2 = \sum_{k=1}^{s(m)-1} \sum_{n=r_{t(k)}+1}^{r_{t(k)+1}} x''^*_n(x_0)x''_n$ (by (4.14)).

By (4.14) it also follows, for each m, that, setting

$$
\begin{aligned}
A_1 &= \sum_{n=1}^{r_{t(s(m))}} [x_n'^*(x')x_n' + x_n''^*(x')x_n''] \\
&= \sum_{n=1}^{r_{t(s(m))}} x_n'^*(x')x_n' + \sum_{n=1}^{r_{t(s(m))+1}} x_n''^*(x')x_n'' \\
&= \sum_{n=1}^{r_{t(s(m))}} x_n'^*(x')x_n' + \sum_{n=1}^{r_{t(1)}} x_n''^*(x')x_n'' + \sum_{k=1}^{s(m)-1} \sum_{n=r_{t(k)+1}+1}^{r_{t(k+1)}} x_n''^*(x')x_n'' \\
&= \sum_{n=1}^{r_{t(s(m))}} x_n'^*(x_0)x_n' + \sum_{n=1}^{r_{t(1)}} x_n''^*(x_0)x_n'' + \sum_{k=1}^{s(m)-1} \sum_{n=r_{t(k)+1}+1}^{r_{t(k+1)}} x_n''^*(x_0)x_n'',
\end{aligned} \tag{4.15}
$$

we have that

$$
\sum_{n=1}^{r_{t(s(m))}} [x_n'^*(x_0)x_n' + x_n''^*(x_0)x_n''] = A_1 + A_2;
$$

$$
\operatorname{dist}(x' - A_1, \operatorname{span}\{\{x_n'\}_{n>r_{t(s(m))}} \cup \{x_n''\}_{n>r_{t(s(m))+1}}\}) = 0
$$

(the first relation follows from the definition of A_2 and from the last equality in the definition of A_1; the second relation follows from the first equality in the definition of A_1 and from the fact that $\|x_n^*\| < H$ for each n; the third relation follows from the second equality in the definition of A_1). We claim that

$$
\operatorname{dist}(x'_{0,m} - A_1, \operatorname{span}\{x_n'\}_{n=r_{t(s(m))}+1}^{r_{t(s(m))+1}})) < \frac{1}{2^{m+2}}. \tag{4.16}
$$

Indeed, $x'_{0,m} - A_1 \in \operatorname{span}\{x_n', x_n''\}_{n=1}^{r_{t(s(m))}}$, hence, by (4.13) (when x is replaced by $x'_{0,m} - A_1$ and m is replaced by $t(s(m))$),

$$
\begin{aligned}
&\operatorname{dist}(x'_{0,m} - A_1, \operatorname{span}\{x_n'\}_{n=r_{t(s(m))}+1}^{r_{t(s(m))+1}})) \\
&\leq \{\operatorname{dist}(x'_{0,m} - A_1, \operatorname{span}\{\{x_n'\}_{n>r_{t(s(m))}} \cup \{x_n''\}_{n>r_{t(s(m))+1}}\})\} \\
&\quad +\{\|x'_{0,m} - A_1\|/[2^{t(s(m))+1}(1 + r_{t(s(m))}H)]\} \\
&\leq \{\|x'_{0,m} - x'\| + \operatorname{dist}(x' - A_1, \operatorname{span}\{\{x_n'\}_{n>r_{t(s(m))}} \cup \{x_n''\}_{n>r_{t(s(m))+1}}\})\} \\
&\quad +\{\|x'_{0,m} - x'\|/[2^{t(s(m))+1}(1 + r_{t(s(m))}H)] + \|x' - A_1\|/[2^{t(s(m))+1}(1 + r_{t(s(m))}H)]\} \\
&< \{\|x'_{0,m} - x'\|\}(\text{by the third relation of (4.15)}) \\
&\quad +\{\|x'_{0,m} - x'\|/2^{t(s(m))} + \|x'\|/2^{t(s(m))}\}(\text{by the second relation of (4.15)}) \\
&< (\text{by (i) and (ii), since } t(s(m)) \geq s(m) > m + 4)\frac{1}{2^{m+3}} + \frac{1}{2^{m+3}} \cdot \frac{1}{2^m} + \frac{1}{2^{m+4}}.
\end{aligned}
$$

By (4.16) there exists $v_m \in \operatorname{span}\{x'_n\}_{n=r_{t(s(m))}+1}^{r_{t(s(m))+1}}$ such that $\|x'_{0,m} - (A_1 + v_m)\| < 1/2^{m+2}$; therefore we can conclude that

$$\Big\|x_0 - \Big\{ \sum_{n=1}^{r_{t(s(m))}} [x_n'^*(x_0)x'_n + x_n''^*(x_0)x''_n] + v_m \Big\}\Big\| = \|\{x'' - A_2\} + \{x'(A_1 + v_m)\}\|$$

(by the first relation of (4.15))

$$\leq \|x'' - A_2\| + \|x' - x'_{0,m}\| + \|x'_{0,m} - (A_1 + v_m)\|$$

$<$ (by (i) and (ii), moreover by what we have specified above) $1/2^{m+1} + 1/2^{m+3} + 1/2^{m+2} < 1/2^m$. This completes the proof of Lemma 4.2.

5. Extension of the uniformly minimal basis with quasi-fixed brackets and permutations

Proof of Theorem II: We start from the M-basis $\{w_n\}$ of Proposition 4.1.

We shall define two new sequences $\{y_n\}$ and $\{z_n\}$, with $\{y_n, y_n^*\} \cup \{z_n, z_n^*\}$ biorthogonal, such that $\{y_n\}$ is a block perturbation of $\{u_n\}$ and $\{y_n, y_n^*\} \cup \{z_n, z_n^*\}$ is a block perturbation of $\{u_n, u_n^*\} \cup \{v_n, v_n^*\}$; we mean that there will be an increasing sequence $\{q_m\}$ of positive integers such that, for each m,

$$\begin{aligned} &\operatorname{span}\{y_n\}_{n=q_m+1}^{q_{m+1}} = \operatorname{span}\{u_n\}_{n=q_m+1}^{q_{m+1}}, \\ (5.1)\qquad &\operatorname{span}\{y_n, z_n\}_{n=q_m+1}^{q_{m+1}} = \operatorname{span}\{u_n, v_n\}_{n=q_m+1}^{q_{m+1}}, \\ &\operatorname{span}\{y_n^*, z_n^*\}_{n=q_m+1}^{q_{m+1}} = \operatorname{span}\{u_n^*, v_n^*\}_{n=q_m+1}^{q_{m+1}}. \end{aligned}$$

Setting $\{y_n, y_n^*\}_{n=1}^{q_0} = \{u_1, u_1^*\}$ and $\{z_n, z_n^*\}_{n=1}^{q_0} = \{v_1, v_1^*\}$, by induction we suppose to have defined, for some $m \geq 0$, $\{y_n, y_n^*\}_{n=1}^{q_m} \cup \{z_n, z_n^*\}_{n=1}^{q_m}$ and we are going to define $\{y_n, y_n^*\}_{n=q_m+1}^{q_{m+1}} \cup \{z_n, z_n^*\}_{n=q_m+1}^{q_{m+1}}$. Unlike the proof of Theorem 1, we do not start from the sequence $\{r_m\}$ of Reference I* of the Introduction, but we prefer to directly define this sequence during the construction.

(5.2)

There exists an integer $r'_{m,0}$ such that

(i) for each x of $\operatorname{span}\{u_n, v_n\}_{n=1}^{q_m} = \operatorname{span}\{y_n, z_n\}_{n=1}^{q_m}$,

$$|\operatorname{dist}(x, \operatorname{span}\{u_n, v_n\}_{n>q_m}) - \operatorname{dist}(x, \operatorname{span}\{\{u_n\}_{n=q_m+1}^{r'_{m,0}} \cup \{v_n\}_{n=q_m+1}^{r'_{m,0}}\})| < \|x\|/\{2^m(2 + 2q_m H)\};$$

(ii) for each x of $\operatorname{span}\{u_n\}_{n=1}^{q_m}$,

$$|\operatorname{dist}(x, \operatorname{span}\{u_n\}_{n>q_m}) - \operatorname{dist}(x, \operatorname{span}\{u_n\}_{n=q_m+1}^{r'_{m,0}})| < \|x\|/\{2^m(2 + q_m H)\}.$$

We start with the elements y_n: Fix i with $1 < i \le r'_{m,0} - q_m$ and assume to have already defined the biorthogonal system

$$\{\{y_{q_m+j}, y^*_{q_m+j}\} \cup \{y_n, y^*_n\}_{n=r'_{m,j-1}+1}^{r'_{m,j}}\}_{j=1}^{i-1};$$

then we are going to define a biorthogonal system

$$\{y_{q_m+i}, y^*_{q_m+i}\} \cup \{y_n, y^*_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i}}$$

(the same definition works also for the first step, that is for $i = 1$). With the $\{w_n\}$ norming, by Reference I* of the Introduction, by (c) of Proposition 4.1 and by the proof of (4.13), there exists an integer $r'_{m,i,0} \ge r'_{m,i-1}$ such that

(5.3)

(i) for each x of $\operatorname{span}\{\{u_n\}_{n=1}^{r'_{m,i-1}} \cup \{v_n\}_{n=1}^{q_m}\}$, $|\operatorname{dist}(x, \operatorname{span}\{\{u_n\}_{n>r'_{m,i-1}}$
$\cup\{v_n\}_{n>r'_{m,i,0}}\}) - \operatorname{dist}(x, \operatorname{span}\{u_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i,0}})|$
$\le \|x\|/\{2^{m+1}[1 + H(r'_{m,i-1} + q_m)]\}$;

(ii) for each x of $\operatorname{span}\{u_n\}_{n=1}^{r'_{m,i-1}}$, $|\operatorname{dist}(x, \operatorname{span}\{u_n\}_{n>r'_{m,i-1}})$
$- \operatorname{dist}(x, \operatorname{span}\{u_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i,0}})|$
$\le \|x\|/\{2^m(2 + Hr'_{m,i-1})\}$.

We point out that, from (5.2) and (5.3), it follows that we have both the properties of (4.13) and the properties of the sequence $\{r_m\}$ of Reference I* of the Introduction. Then setting

(5.4)
$S'_{m,i} = 2^{m+6}H(r'_{m,i-1} + q_m)$, we choose a sequence
$\{w'_{m,i,s}\}_{s=1}^{L'_{m,i}}$, with $w'_{m,i,s} = \sum_{n=r'_{m,i-1}+1}^{r'_{m,i,0}} a_{m,i,s,n}u_n$
for $1 \le s \le L'_{m,i}$, which is $(1/S'_{m,i})$-dense in the ball of radius $S'_{m,i}$ of span $\{u_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i,0}}$.

Choose now two positive integers $A'(m,i)$ and $r'_{m,i}$ as follows:

let $A'(m,i)$ denote the first integer $\ge 8H \cdot S'^3_{m,i}$,
such that there exists another integer $r'_{m,i}$,
such that the following properties hold:

$$(5.5)\quad \begin{aligned}&\text{(a) } \{u_n\}_{n=r'_{m,i,0}+1}^{r'_{m,i}} = \{u_{0,m,i,n}\}_{n=1}^{T'_{m,i}} \cup \{u_{0,m,i,0,n}\}_{n=1}^{T'_{m,i,0}},\\ &\{u_{0,m,i,n}\}_{n=1}^{T'_{m,i}} = \{\{\{u_{f,j,n}\}_{n=1}^{2^f}\}_{j=1}^{R_f}\}_{f=A'(m,i)+1}^{A'(m,i)+L'_{m,i}\cdot S'^2_{m,i}}\\ &= \{\{\{\{u_{f(m,i,s,k),j,n}\}_{n=1}^{2^{f(m,i,s,k)}}\}_{j=1}^{R_{f(m,i,s,k)}}\}_{k=1}^{S'^2_{m,i}}\}_{s=1}^{L'_{m,i}}\\ &\text{and } T'_{m,i,0} > (1 + r'_{m,i,0} - r'_{m,i-1})4^{4H^2 S'^3_{m,i} T'_{m,i}};\\ &\text{(b) } \{v_n\}_{n>q_m} \supset \{\{v_{f,0,n}\}_{n=1}^{Q_{f,0}}\}_{f=A'(m,i)+1}^{A'(m,i)+L'_{m,i}\cdot S'^2_{m,i}}\end{aligned}$$

(we use sequence $\{\{u_{f,j,n}\}_{n=1}^{2^f}\}_{j=1}^{R_f} \cup \{v_{f,0,n}\}_{n=1}^{Q_{f,0}}$ of Proposition 4.1). We then set

$$y'_{q_m+i} = u_{q_m+i}/S'^2_{m,i} - \sum_{s=1}^{L'_{m,i}} \sum_{k=1}^{S'^2_{m,i}} \frac{k}{S'^2_{m,i}} \sum_{j=1}^{R_{f(m,i,s,k)}} u_{f(m,i,s,k),j,1}$$

and

$$y'^*_{q_m+1} = S'^2_{m,i} u^*_{q_m+i};$$

moreover, for each set of indices $\{k, s, j\}$, with $1 \le k \le S'^2_{m,i}$, $1 \le s \le L'_{m,i}$ and $1 \le j \le R_{f(m,i,s,k)}$, we set $y'_{f(m,i,s,k),j,1} = u_{f(m,i,s,k),j,1} + w'_{m,i,s}$ and $y'^*_{f(m,i,s,k),j,1} = u^*_{f(m,i,s,k),j,1} + k \cdot u^*_{q_m+i}$, while

$$(5.6)\quad \begin{aligned}&y'_{f(m,i,s,k),j,n} = u_{f(m,i,s,k),j,n} \quad \text{and} \quad y'^*_{f(m,i,s,k),j,n} = u^*_{f(m,i,s,k),j,n}\\ &\quad \text{for } 2 \le n \le 2^{f(m,i,s,k)};\\ &y'_{0,m,i,0,n} = u_{0,m,i,0,n} \quad \text{and} \quad y'^*_{0,m,i,0,n} = u^*_{0,m,i,0,n}\\ &\quad \text{for } 1 \le n \le T'_{m,i,0};\end{aligned}$$

while $y'_n = u_n$ for $r'_{m,i-1} + 1 \le n \le r'_{m,i,0}$. By (5.4) and (5.6), by means of a formula analogous to (3.8), there exists

$$\begin{gathered}\{y'_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i,0}} \text{ in span } \{\{u^*_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i,0}} \cup u^*_{q_m+i} \cup \{u^*_{0,m,i,n}\}_{n=1}^{T'_{m,i}}\}\\ \text{such that } \{y'_{q_m+i}, y'^*_{q_m+i}\} \cup \{y'_n, y'^*_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i,0}} \cup \{y'_{0,m,i,n}, y'^*_{0,m,i,n}\}_{n=1}^{T'_{m,i}} \text{ is}\\ \text{biorthogonal with span}\{y'^*_{q_m+i} \cup \{y'^*_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i,0}} \cup \{y'^*_{0,m,i,n}\}_{n=1}^{T'_{m,i}}\}\\ = \text{span}\{u^*_{q_m+i} \cup \{u^*_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i,0}} \cup \{u^*_{0,m,i,n}\}_{n=1}^{T'_{m,i}}\}.\end{gathered}$$

Proceeding as in Theorem I, through (3.9) up to (3.10), we pass from

$$\{y'_{q_m+i}, y'^*_{q_m+i}\} \cup \{y'_n, y'^*_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i}} \text{ to } \{y_{q_m+i}, y^*_{q_m+i}\} \cup \{y_n, y^*_n\}_{n=r'_{m,i-1}+1}^{r'_{m,i}}$$

with $\|y_{q_m+i}\| = 1$ and $\|y^*_{q_m+i}\| < 3H$, $\|y_n\| = 1$ and $\|y^*_n\| < 3H$ for $r'_{m,i-1} + 1 \leq n \leq r'_{m,i}$. Proceeding in this way up to $i = r'_{m,0} - q_m$ and setting

$$r'''_{m,0} = r'_{m,i} \text{ for } i = r'_{m,0} - q_m \text{ we have that:}$$

(5.7)
$$\operatorname{span}\{y_n\}_{n=q_m+1}^{r'''_{m,0}} = \operatorname{span}\{u_n\}_{n=q_m+1}^{r'''_{m,0}} \text{ and}$$
$$\operatorname{span}\{y^*_n\}_{n=q_m+1}^{r'''_{m,0}} = \operatorname{span}\{u^*_n\}_{n=q_m+1}^{r'''_{m,0}}.$$

We give an analogous construction for the elements z_n:

At first we pass from $r'''_{m,0}$ to $r''_{m,0}$ such that (see (b) of (5.5))

(5.8)
$$y_n = u_n \text{ and } y^*_n = u^*_n \text{ for } r'''_{m,0} + 1 \leq n \leq r''_{m,0},$$
$$\{v_n\}_{n=q_m+1}^{r''_{m,0}} \supset \{\{\{v_{f,0,n}\}_{n=1}^{Q_{f,0}}\}_{f=A'(m,i)+1}^{A'(m,i)+L'_{m,i}\cdot S'^2_{m,i}}\}_{i=1}^{r'_{m,0}-q_m}.$$

Again we proceed by induction and we only describe the general step. That is, fix i with $1 < i \leq r''_{m,0} - q_m$ and suppose we have already defined the biorthogonal system $\{\{z_{q_m+j}, z^*_{q_m+j}\} \cup \{y_n, y^*_n\}_{n=r''_{m,j-1}+1}^{r''_{m,j}} \cup \{z_n, z^*_n\}_{n=r''_{m,j-1}+1}^{r''_{m,j}}\}_{j=1}^{i-1}$; then we are going to define a biorthogonal system $\{z_{q_m+i}, z^*_{q_m+i}\} \cup \{y_n, y^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i}} \cup \{z_n, z^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i}}$ (again the construction works also for the first step, that is for $i = 1$).

There exists an integer $r''_{m,i,0} \geq r''_{m,i-1}$ such that,

(5.9)
$$\text{for each } x \text{ of span } \{\{u_n\}_{n=1}^{r''_{m,i-1}} \cup \{v_n\}_{n=1}^{r''_{m,i-1}}\},$$
$$|\operatorname{dist}(x, \operatorname{span}\{\{u_n\}_{n>r''_{m,i-1}} \cup \{v_n\}_{n>r''_{m,i-1}}\})$$
$$-\operatorname{dist}(x, \operatorname{span}\{\{u_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}} \cup \{v_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}}\})| \leq \|x\|/\{2^{m+1}(2 + Hr''_{m,i-1})\}.$$

Setting

$$S''_{m,i} = 2^{m+7}Hr''_{m,i-1}, \text{ we choose a sequence}$$

(5.10)
$$\{w''_{m,i,s}\}_{s=1}^{L''_{m,i}}, \text{ with } w''_{m,i,s} = \sum_{n=r''_{m,i-1}+1}^{r''_{m,i,0}} (b_{m,i,s,n}u_n + c_{m,i,s,n}v_n)$$
for $1 \leq s \leq L''_{m,i}$, which is $(1/S''_{m,i})$-dense in the ball of radius $S''_{m,i}$ of span $\{\{u_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}} \cup \{v_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}}\}$.

After, we choose two integers $A''(m,i)$ and $r''_{m,i}$ as follows:

$$\begin{aligned}
&A''(m,i) \text{ is the first integer } \geq 8H\cdot S''^3_{m,i} \text{ such that}\\
&\text{there exists another integer } r''_{m,i} \text{ such that}\\
&\{v_n\}_{n=r''_{m,i,0}+1}^{r''_{m,i}} = \{v_{0,m,i,n}\}_{n=1}^{T''_{m,i}} \cup \{v_{0,m,i,0,n}\}_{n=1}^{T''_{m,i,0}},\\
&\{v_{0,m,i,n}\}_{n=1}^{T''_{m,i}} = \{\{\{v_{f,j,n}\}_{n=1}^{2^f}\}_{j=1}^{T_f}\}_{f=A''(m,i)+1}^{A''(m,i)+L''_{m,i}\cdot S''^2_{m,i}}\\
&= \{\{\{\{v_{f(m,i,s,k),j,n}\}_{n=1}^{2^{f(m,i,s,k)}}\}_{j=1}^{T_{f(m,i,s,k)}}\}_{k=1}^{S''^2_{m,i}}\}_{s=1}^{L''_{m,i}}\\
&\text{and } T''_{m,i,0} > (1+r''_{m,i,0}-r''_{m,i-1})4^{4H^2 S''^3_{m,i}T''_{m,i}}
\end{aligned}\tag{5.11}$$

(we use the sequences $\{\{v_{f,j,n}\}_{n=1}^{2^f}\}_{j=1}^{T_f}$ of Proposition 4.1). Then we set

$$z'_{q_m+i} = v_{q_m+i}/S''^2_{m,i} - \sum_{s=1}^{L''_{m,i}} \sum_{k=1}^{S''^2_{m,j}} \frac{k}{S''^2_{m,i}} \sum_{j=1}^{T_{f(m,i,s,k)}} v_{f(m,i,s,k),j,1}$$

and $z'^*_{q_m+i} = S''^2_{m,i} v^*_{q_m+i}$; moreover, for each set of indices

$\{k,s,j\}$, with $1\leq k\leq S''^2_{m,i}$, $1\leq s\leq L''_{m,i}$ and $1\leq j\leq T_{f(m,i,s,k)}$,

$$\begin{aligned}
&\text{we set } z'_{f(m,i,s,k),j,1} = v_{f(m,i,s,k),j,1} + w''_{m,i,s} \text{ and } z'^*_{f(m,i,s,k),j,1}\\
&= v^*_{f(m,i,s,k),j,1} + k\cdot v^*_{q_m+i}, \text{ while } z'_{f(m,i,s,k),j,n} = v_{f(m,i,s,k),j,n}\\
&\text{and } z'^*_{f(m,i,s,k),j,n} = v^*_{f(m,i,s,k),j,n} \text{ for } 2\leq n\leq 2^{f(m,i,s,k)};
\end{aligned}\tag{5.12}$$

$$z'_{0,m,i,0,n} = v_{0,m,i,0,n} \text{ and } z'^*_{0,m,i,0,n} = v^*_{0,m,i,0,n} \text{ for } 1\leq n\leq T''_{m,i,0};$$

$$z'_n = v_n \text{ for } r''_{m,i-1}+1\leq n\leq r''_{m,i,0}; \text{ finally}$$

$$y'_n = u_n \text{ for } r''_{m,i-1}+1\leq n\leq r''_{m,i} \text{ and } y'^*_n = u^*_n \text{ for } r''_{m,i,0}+1\leq n\leq r''_{m,i}.$$

By (5.10) and (5.12), by means of a formula analogous to (3.8), there exists

$$\{y'^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}} \cup \{z'^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}} \text{ in } X^* \text{ such that}$$

$$\{z'_{q_m+i}, z'^*_{q_m+i}\} \cup \{y'_n, y'^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}} \cup \{z'_n, z'^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}}$$

$$\cup\{z'_{0,m,i,n}, z'^*_{0,m,i,n}\}_{n=1}^{T''_{m,i}} \text{ is biorthogonal, with}$$

$$\operatorname{span}\{z'^*_{q_m+i} \cup \{y'^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}} \cup \{z'^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}} \cup \{z'^*_{0,m,i,n}\}_{n=1}^{T''_{m,i}}\} =$$

$$\operatorname{span}\{v^*_{q_m+i}\cup\{u^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}}\cup\{v^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i,0}}\cup\{v^*_{0,m,i,n}\}_{n=1}^{T''_{m,i}}\}.$$

Proceeding as for Theorem I, from (3.8) to (3.9), by (5.10), (5.11) and (5.12) we obtain that

$$\begin{gathered}\|z'_{q_m+i}\|\cdot\|z'^*_{q_m+i}\|<2HS''^2_{m,i}T''_{m,i};\text{ moreover, for each set of}\\ \text{indices }\{s,k,j\}\text{ with }1\le k\le S''^2_{m,i},1\le s\le L''_{m,i}\text{ and }1\le j\le T_{f(m,i,s,k)},\\ \|z'_{f(m,i,s,k),j,1}\|\cdot\|z'^*_{f(m,i,s,k),j,1}\|<4HS''^3_{m,i};\\ \text{while, for each }n\text{ with }r''_{m,i-1}+1\le n\le r''_{m,i,0},\\ \|z'^*_n\|<H+HS''_{m,i}\cdot 2HS''^2_{m,i}\cdot T''_{m,i}<4H^2S''^3_{m,i}T''_{m,i}\text{ and }\|y'^*_n\|<4H^2S''^3_{m,i}T''_{m,i}.\end{gathered}\tag{5.13}$$

By Reference II* of the Introduction, and by (5.11), (5.12) and (5.13), there exists a block perturbation $\{z_{q_m+i},z^*_{q_m+i}\}\cup\{z_n,z^*_n\}_{r''_{m,i-1}+1}^{r''_{m,i,0}}\cup\{z_{0,m,i,0,n},z^*_{0,m,i,0,n}\}_{n=1}^{T''_{m,i,0}}$ of $\{z'_{q_m+i},z'^*_{q_m+i}\}\cup\{z'_n,z'^*_n\}_{r''_{m,i-1}+1}^{r''_{m,i,0}}\cup\{z'_{0,m,i,0,n},z'^*_{0,m,i,0,n}\}_{n=1}^{T''_{m,i,0}}$ with $\|z_{q_m+i}\|=1$ and $\|z^*_{q_m+i}\|<3H$, $\|z_n\|=1$ and $\|z^*_n\|<3H$ for $r''_{m,i-1}+1\le n\le r''_{m,i,0}$, $\|z_{0,m,i,0,n}\|=1$ and $\|z^*_{0,m,i,0,n}\|<3H$ for $1\le n\le T''_{m,i,0}$. Again by (5.12) and (5.13), since by (5.11),

$$r''_{m,i}-r''_{m,i,0}>T''_{m,i,0}>(r''_{m,i,0}-r''_{m,i-1})4^{4H^2S''^3_{m,i}T''_{m,i}},$$

there exists a block perturbation $\{y_n,y^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i}}$ of $\{y'_n,y'^*_n\}_{n=r''_{m,i-1}+1}^{r''_{m,i}}$ with $\|y_n\|=1$ and $\|y^*_n\|<3H$ for $r'_{m,0,i-1}+1\le n\le r'_{m,0,i}$. Finally by (5.12) and (5.13), for each set of indices $\{k,s,j\}$, with $1\le k\le S''^2_{m,i}$, $1\le s\le L''_{m,i}$ and $1\le j\le T_{f(m,i,s,k)}$, since by (5.11), $f(m,i,s,k)>A''(mi)\ge 8H\cdot S''^3_{m,i}$, there exists a block perturbation $\{z_{f(m,i,s,k),j,n},z^*_{f(m,i,s,k),j,n}\}_{n=1}^{2^{f(m,i,s,k)}}$ of $\{z'_{f(m,i,s,k),j,n},z'^*_{f(m,i,s,k),j,n}\}_{n=1}^{2^{f(m,i,s,k)}}$ with $\|z_{f(m,i,s,k),j,n}\|=1$ and $z^*_{f(m,i,s,k),j,n}\|<3H$ for $1\le n\le 2^{f(m,i,s,k)}$.

Proceeding in this way up to $i=r''_{m,0}-q_m$ we have that,

$$\begin{gathered}\text{setting }q_{m+1}=r''_{m,r''_{m,0}-q_m},\text{ then: span }\{y_n\}_{n=r''_{m,0}+1}^{q_{m+1}}\\ =\operatorname{span}\{u_n\}_{n=r''_{m,0}+1}^{q_{m+1}};\operatorname{span}\{\{z_n\}_{n=q_m+1}^{q_{m+1}}\cup\{y_n\}_{n=r''_{m,0}+1}^{q_{m+1}}\}\\ =\operatorname{span}\{\{v_n\}_{n=q_m+1}^{q_{m+1}}\cup\{u_n\}_{n=r''_{m,0}+1}^{q_{m+1}}\}\text{ and span }\{\{z^*_n\}_{n=q_m+1}^{q_{m+1}}\\ \cup\{y^*_n\}_{n=r''_{m,0}+1}^{q_{m+1}}\}=\operatorname{span}\{\{v^*_n\}_{n=q_m+1}^{q_{m+1}}\cup\{u^*_n\}_{n=r''_{m,0}+1}^{q_{m+1}}\}.\end{gathered}\tag{5.14}$$

By (5.7), (5.8) and (5.14) it follows that (5.1) holds. Now the construction has been completed and we proceed to verify the assertion; we begin directly with the proof that $\{x_n\} = \{y_n\} \cup \{z_n\}$ is a basis with quasi-fixed brackets and permutations of X.

Letting then x_0 be an element of X with $\|x_0\| = 1$, we have that $\{m\} = \{m(t)\} \cup \{m'(t)\} \cup \{m''(t)\}$ where one or two of these subsequences can be finite or void) such that, for each t:

(5.15)

(a) $|v^*_{q_{m(t)}+i(t)}(x_0)| > 1/S''^2_{m(t),i(t)}$ for $i(t)$ with $1 \le i(t) \le r''_{m(t),0} - q_{m(t)}$ and $|v^*_{q_{m(t)}+j}(x_0)| \le 1/S''^2_{m(t),j}$ for $i(t)+1 \le j \le r''_{m(t),0} - q_{m(t)}$;

(b) $|v^*_{q_{m'(t)}+j}(x_0)| \le 1/S''^2_{m'(t),j}$ for $1 \le j \le r''_{m'(t),0} - q_{m'(t)}$, but $|u^*_{q_{m'(t)}+i(t)}(x_0)| > 1/S'^2_{m'(t),i(t)}$ for $i(t)$ with $1 \le i(t) \le r'_{m'(t),0} - q_{m'(t)}$ and $|u^*_{q_{m'(t)}+j}(x_0)| \le 1/S'^2_{m'(t),j}$ for $i(t)+1 \le j \le r'_{m'(t),0} - q_{m'(t)}$;

(c) $|v^*_{q_{m''(t)}+j}(x_0)| \le 1/S''^2_{m''(t),j}$ for $1 \le j \le r''_{m''(t),0} - q_{m''(t)}$ and $|u^*_{q_{m''(t)}+j}(x_0)| \le 1/S'^2_{m''(t),j}$ for $1 \le j \le r'_{m''(t),0} - q_{m''(t)}$.

Again we consider separately the three subsequences:

(A) Suppose that $\{m(t)\}$ of (a) of (5.15) is infinite.

Starting from (a) of (5.15), we can follow step by step the procedure of the proof of (A) of Theorem I, from (3.11) up to (3.18). We point out that, by means of (5.9) and following the proof of (3.14), there exists a subsequence $\{t''\}$ of $\{t\}$ such that, for each t and for each $t' \ge t''$, there exists

$$w_{t'} \in \operatorname{span}\{\{u_n\}_{n=r''_{m(t',i(t')-1}+1}^{r''_{m(t'),i(t'),0}} \cup \{v_n\}_{n=r''_{m(t'),i(t')-1}+1}^{r''_{m(t'),i(t'),0}}\}$$

such that, analogously to (3.14),

$$\left\| x_0 - \left\{ \sum_{n=1}^{r''_{m(t'),i(t')-1}} y_n^*(x_0) y_n + \sum_{n=1}^{q_{m(t')}+i(t')-1} z_n^*(x_0) z_n + \sum_{n=r''_{m(t'),0}+1}^{r''_{m(t'),i(t')-1}} z_n^*(x_0) z_n + w_{t'} \right\} \right\| < 1/2^{t+1}.$$

Then fix t and $t' \ge t''$: We can choose an index $k(t')$, with $1 \le k(t') \le S''^2_{m(t'),i(t')}$, such that a property analogous to (3.16) holds. Moreover, by (5.10) and by means of the same procedure of Theorem I, we can choose an index $s(t')$, with $1 \le s(t') \le L''_{m(t'),i(t')}$, such that a property analogous to (3.17) holds. At this point we consider the fixed index $f(m(t'), i(t'), s(t'), k(t'))$ and, by means of (a) of Proposition 4.1 (see (5.11)), we can choose an index $j(t')$, with $1 \le j(t') \le T_{f(m(t'),i(t'),s(t'),k(t'))}$, such that, analogously to (3.15),

$$\Big\|x_0 - \Big\{ \sum_{n=1}^{r''_{m(t'),i(t')-1}} y_n^*(x_0)y_n + \sum_{n=1}^{q_{m(t')}+i(t')-1} z_n^*(x_0)z_n + \sum_{n=r''_{m(t'),0}+1}^{r''_{m(t'),i(t')-1}} z_n^*(x_0)z_n +$$

$$\sum_{n=1}^{2^{f(m(t'),i(t'),s(t'),k(t'))}} z^*_{f(m(t'),i(t'),s(t'),k(t')),j(t'),n}(x_0) z_{f(m(t'),i(t'),s(t'),k(t')),j(t'),n}\Big\}\Big\| < 1/2^t.$$

Therefore there exist for each t a permutation

$\{\pi''(n)\}_{n=q_{m(t)}+1}^{q_{m(t)+1}}$ of $\{n\}_{n=q_{m(t)}+1}^{q_{m(t)+1}}$, two integers $q'_{0,m(t)}$ and $q''_{0,m(t)}$ with $q_{m(t)}+1 \le q'_{0,m(t)}, q''_{0,m(t)} \le q_{m(t)+1}$ and a positive $\varepsilon'_{m(t)}$ so that $\varepsilon'_{m(t)} \to 0$ with t,

(5.16) such that $\Big\|x_0 - \Big\{ \sum_{n=1}^{q_{m(t)}} [y_n^*(x_0)y_n + z_n^*(x_0)z_n]$

$$+ \sum_{n=q_{m(t)}+1}^{q'_{0,m(t)}} y_n^*(x_0)y_n + \sum_{n=q_{m(t)}+1}^{q''_{0,m(t)}} z^*_{\pi''(n)}(x_0) z_{\pi''(n)} \Big\}\Big\| < \varepsilon'_{m(t)}.$$

(B) Suppose that $\{m'(t)\}$ of (b) of (5.15) is infinite (this can happen in particular if $x_0 \in Y$, therefore the cases (B) and (C) include also the proof that $\{y_n\}$ is a basis with quasi-fixed brackets and permutations of Y; for this proof in (5.2) and (5.3) we have only to use (ii). Again, starting from the second part of (b) of (5.15), we can follow step by step the procedure of the proof of (A) of Theorem I, from (3.11) up to (3.18). We point out that (i) of (5.3) is similar to (4.13) of Lemma 4.2; moreover, also the hypothesis on x_0 of Lemma 4.2 holds since, by the first part of (b) of (5.15), for each t, since by (5.7) and (5.8), $r'_{m'(t),i(t),0} < r''_{m'(t),0}$, moreover by (5.10), we have that

$$\sum_{n=q_{m'(t)}+1}^{r'_{m'(t),i(t),0}} |v_n^*(x_0)| < \sum_{n=q_{m'(t)}+1}^{r'_{m'(t),i(t),0}} \frac{1}{S''^2_{m'(t),n-q_{m'(t)}}} < \frac{1}{2^{m'(t)}};$$

hence the statement of Lemma 4.2 holds and, again, there exists a subsequence $\{t''\}$ of $\{t\}$ such that, for each t and for each $t' \ge t''$, there is

$$w_{t'} \in \operatorname{span}\{u_n\}_{n=r'_{m(t'),i(t')-1}+1}^{r'_{m(t'),i(t'),0}}$$

such that, analogously to (3.14),

$$\left\|x_0 - \left\{ \sum_{n=1}^{q_{m'(t')}+i(t')-1} y_n^*(x_0)y_n + \sum_{n=r'_{m'(t'),0}+1}^{r'_{m'(t'),i(t')-1}} y_n^*(x_0)y_n + \sum_{n=1}^{q_{m'(t')}} z_n^*(x_0)z_n + w_{t'} \right\}\right\| < \frac{1}{2^{t+1}}.$$

Hence fix t and $t' \geq t''$: We can choose an index $k(t')$, with $1 \leq k(t') \leq S'^2_{m'(t'),i(t')}$, such that a property analogous to (3.16) holds; moreover, by (5.4) and by means of the same procedure of Theorem I, we can choose an index $s(t')$, with $1 \leq s(t') \leq L'_{m'(t'),i(t')}$, such that a property analogous to (3.17) holds. At this point we consider the fixed index $f(m'(t'),i(t'),s(t'),k(t'))$ and, by (5.8), we have that

$$\{v_n\}_{n=q_{m'(t')}+1}^{r''_{m'(t'),0}} \supset \{v_{f(m'(t'),i(t'),s(t'),k(t')),0,n}\}_{n=1}^{Q_{f(m'(t'),i(t'),s(t'),k(t')),0}};$$

hence, by the first part of (b) of (5.15) and by (5.10),

$$\sum_{n=1}^{Q_{f(m'(t'),i(t'),s(t'),k(t')),0}} |v^*_{f(m'(t'),i(t'),s(t'),k(t')),0,n}(x_0)| \leq \frac{1}{S''^2_{m'(t'),1}} \cdot Q_{f(m'(t'),i(t'),s(t'),k(t')),0} < \frac{1}{4};$$

that is, the hypothesis of (b) of Proposition 4.1 is verified and the statement holds. Then we can choose an index $j(t')$, with $1 \leq j(t') \leq R_{f(m'(t'),i(t'),s(t'),k(t'))}$, such that, analogously to (3.15),

$$\Bigg\|x_0 - \Bigg\{ \sum_{n=1}^{q_{m'(t')}+i(t')-1} |y_n^*(x_0)y_n + \sum_{n=r'_{m'(t'),0}+1}^{r'_{m'(t'),i(t')-1}} y_n^*(x_0)y_n + \sum_{n=1}^{q_{m'(t')}} z_n^*(x_0)z_n + \sum_{n=1}^{2^{f(m'(t'),i(t'),s(t'),k(t'))}} y^*_{f(m'(t'),i(t'),s(t'),k(t')),j(t'),n}(x_0) \times y_{f(m'(t'),i(t'),s(t'),k(t')),j(t'),n}\Bigg\}\Bigg\| < \frac{1}{2^t}.$$

Therefore again there exist for each t a permutation

(5.17)
$\{\pi'(n)\}_{n=q_{m'(t)}+1}^{q_{m'(t)+1}}$ of $\{n\}_{n=q_{m'(t)}+1}^{q_{m'(t)+1}}$, an integer $q'_{0,m'(t)}$ with $q_{m'(t)}+1 \leq q'_{0,m'(t)} \leq q_{m'(t)+1}$ and a positive $\varepsilon'_{m'(t)} \to 0$ with t, such that

$$\left\|x_0 - \left\{ \sum_{n=1}^{q_{m'(t)}} [y_n^*(x_0)y_n + z_n^*(x_0)z_n] + \sum_{n=q_{m'(t)}+1}^{q'_{0,m'(t)}} y^*_{\pi'(n)}(x_0)y_{\pi'(n)} \right\}\right\| < \varepsilon'_{m'(t)}.$$

(C) Finally, suppose that $\{m''(t)\}$ of (c) of (5.15) is infinite.

By (c) of (5.15), by (i) of (5.2) for the definition of $r'_{m''(t),0}$, by (i) of (5.2) and of (5.3), by (5.7), (5.8), (5.9) and (5.14) for the definitions of $r''_{m''(t),0}$ and of q_{m+1}, and also by (5.4) and (5.10), we have the following two facts: the first one is that

$$\left\| \sum_{n=q_{m''(t)}+1}^{r'_{m''(t),0}} [u_n^*(x_0)u_n + v_n^*(x_0)v_n] \right\| \le \sum_{n=q_{m''(t)}+1}^{r'_{m''(t),0}} |u_n^*(x_0)| + \sum_{n=q_{m''(t)}+1}^{r''_{m''(t),0}} |v_n^*(x_0)| < \frac{2}{2^{m''(t)}} \quad \text{for each } t;$$

while the second fact is that, setting $r_{2m} = q_m$ and $r_{2m+1} = r'_{m,0}$ for each m, by construction this new sequence $\{r_m\}$ has the same properties as the sequence $\{r_m\}$ of Reference I* of the Introduction. Therefore, since $\{u_n\} \cup \{v_n\}$ of Proposition 4.1 is norming, by the second part of Reference I* of the Introduction and by the proof of (3.19), setting $m''(0) = q_0 = 0$, we have

$$x_0 = \sum_{t=0}^{\infty} \sum_{n=q_{m''(t)}+1}^{q_{m''(t+1)}} [y_n^*(x_0)y_n + z_n^*(x_0)z_n];$$

that is there exists, for each t, a positive $\varepsilon'_{m''(t)}$, so that $\varepsilon'_{m''(t)} \to 0$ with t, such that

$$\left\| x_0 - \sum_{n=1}^{q_{m''(t)}} [y_n^*(x_0)y_n + z_n^*(x_0)z_n] \right\| < \varepsilon'_{m''(t)}. \tag{5.18}$$

Again, our aim now is to show that, for each positive integer i,

setting $q(i) = q_{2i}$, there exist two permutations
$\{\pi'(n)\}_{n=q(i)+1}^{q(i+1)}$ and $\{\pi''(n)\}_{n=q(i)+1}^{q(i+1)}$ of $\{n\}_{n=q(i)+1}^{q(i+1)}$,
two integers $q'(0,i)$ and $q''(0,i)$, with $q(i)+1 \le q'(0,i), q''(0,i) \le q(i+1)$,
and a positive number $\varepsilon(i)$ so that $\varepsilon(i) \to 0$ with i, such that

$$\left\| x_0 - \left\{ \sum_{n=1}^{q(i)} [y_n^*(x_0)y_n + z_n^*(x_0)z_n] + \sum_{n=q(i)+1}^{q'(0,i)} y_{\pi'(n)}^*(x_0)y_{\pi'(n)} + \sum_{n=q(i)+1}^{q''(0,i)} z_{\pi''(n)}^*(x_0)z_{\pi''(n)} \right\} \right\| < \varepsilon(i); \tag{5.19}$$

that is, if $q'(0,0) = q''(0,0) = 0$,

$$x_0 = \sum_{i=0}^{\infty} \Big[\sum_{n=q'(0,i)+1}^{q'(0,i+1)} y_{\pi'(n)}^*(x_0)y_{\pi'(n)} + \sum_{n=q''(0,i)+1}^{q''(0,i+1)} z_{\pi''(n)}^*(x_0)z_{\pi''(n)} \Big].$$

Indeed, setting for each i,

$$\{x_n, x_n^*\}_{n=2q(i)+1}^{q(i+1)+q(i)} = \{y_n, y_n^*\}_{n=q(i)+1}^{q(i+1)}$$

and

$$\{x_n, x_n^*\}_{n=q(i+1)+q(i)+1}^{2q(i+1)} = \{z_n, z_n^*\}_{n=q(i)+1}^{q(i+1)},$$
$$q(0,i) = q'(0,i) = q''(0,i),$$
$$\{\pi(n)\}_{n=2q(i)+1}^{q(0,i)} = \{q(i) + \pi'(n)\}_{n=q(i)+1}^{q'(0,i)} \cup \{q(i+1) + \pi''(n)\}_{n=q(i)+1}^{q''(0,i)}$$

and

$$\{\pi(n)\}_{n=q(0,i)+1}^{2q(i+1)} = \{q(i) + \pi'(n)\}_{n=q'(0,i)+1}^{q(i+1)} \cup \{q(i+1) + \pi''(n)\}_{n=q''(0,i)+1}^{q(i+1)},$$

it will follow that

$$x_0 = \sum_{i=0}^{\infty} \sum_{n=q(0,i)+1}^{q(0,i+1)} x_{\pi(n)}^*(x_0)x_{\pi(n)} \quad \text{with } 2q(i)+1 \le q(0,i) \le 2q(i+1)$$

for each i; that is $\{x_n\}$ verifies the properties of (D_4) of the Introduction.

Then, for a fixed $q(i) = q_{2i}$, we have one of the following three possibilities:

(i) $2i+1 = m(t)$ for some t: in this case, by (5.16) we obtain (5.19)

$$\text{for } q'(0,i) = q'_{0,m(t)} \text{ and } \{\pi'(n)\}_{n=q(i)+1}^{q'(0,i)} = \{n\}_{n=q(i)+1}^{q'_{0,m(t)}},$$
$$\text{moreover} \quad q''(0,i) = q''_{0,m(t)} \quad \text{and}$$
$$\{\pi''(n)\}_{n=q(i)+1}^{q''(0,i)} = \{n\}_{n=q(i)+1}^{q_{m(t)}} \cup \{\pi''(n)\}_{n=q_{m(t)}+1}^{q''_{0,m(t)}}, \quad \varepsilon(i) = \varepsilon'_{m(t)};$$

(ii) $2i+1 = m'(t)$ for some t, then by (5.17) we obtain (5.19) for

$$q'(0,i) = q'_{0,m'(t)} \text{ and } \{\pi'(n)\}_{n=q(i)+1}^{q'(0,i)} = \{n\}_{n=q(i)+1}^{q_{m'(t)}} \cup \{\pi'(n)\}_{n=q_{m'(t)}+1}^{q'_{0,m'(t)}},$$
$$\text{moreover} \quad q''(0,i) = q_{m'(t)} \quad \text{and}$$
$$\{\pi''(n)\}_{n=q(i)+1}^{q''(0,i)} = \{n\}_{n=q(i)+1}^{q_{m'(t)}}, \quad \varepsilon(i) = \varepsilon'_{m'(t)};$$

(iii) $2i+1 = m''(t)$ for some t, then by (5.18) we obtain (5.19) for

$$q'(0,i) = q''(0,i) = q_{m''(t)} \quad \text{and}$$
$$\{\pi'(n)\}_{n=q(i)+1}^{q'(0,i)} = \{\pi''(n)\}_{n=q(i)+1}^{q''(0,i)} = \{n\}_{n=q(i)+1}^{q_{m''(t)}}, \quad \varepsilon(i) = \varepsilon'_{m''(t)};$$

hence (5.19) is proved. This completes the proof of Theorem II.

References

[1] G. A. Aleksandrov, *Strong M-bases and equivalent norms in nonseparable Banach spaces*, Godishnik na Visshite Uchebn. Zavedeniya Prilozhna Matematika **19** (1983), 31–44 (Russian).

[2] G. A. Aleksandrov and A. N. Plichko, *The connections between strong M-bases and equivalent locally uniformly convex norms in Banach space*, Compte Rendu Academie Bulgare Science **40** (1987), 15–16 (Russian).

[3] G. A. Aleksandrov and A. N. Plichko, *Relations between strong and norming M-basis in non-separable Banach spaces*, to appear.

[4] S. Banach, *Theorie des opérations linéaires*, Monografie Mat. 1, Waszawa, 1932.

[5] J. Dyer, *Generalized Markushevich bases*, Israel Journal Mathematics **7** (1969), 51–68.

[6] P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Mathematica **130** (1973), 309–317.

[7] W. P. Fonf, *Operator bases and generalized summation bases*, Doklady Akademii Nauk USSR **A11** (1986), 16–18 (Russian).

[8] D. P. Giesy, *On a convexity condition in normed linear spaces*, Transactions of the Amererican Mathematical Society **125** (1966), 114–146.

[9] D. A. Goodner, *Projections in normed linear spaces*, Transactions of the American Mathematical Society **69** (1950), 89–108.

[10] B. V. Godun, *On the bounded and non-bounded fundamental biorthogonal systems in Banach spaces*, Siberian Mathematical Journal **23** (1982), 190–193 (Russian).

[11] B. V. Godun, *Quasi-complementation and minimal systems in l_∞*, Matematicheskie Zametki **36** (1984), 117–121 (Russian).

[12] B. V. Godun, *A special class of Banach spaces*, Matematicheskie Zametki **37** (1985), 391–398 (Russian).

[13] B. V. Godun and S. L. Troyanski, *Renorming Banach spaces with fundamental biorthogonal systems*, Contemporary Mathematics **144** (1993), 119–126.

[14] B. V. Godun, B. L. Lin and S. L. Troyanski, *On Auerbach bases*, Contemporary Mathematics **144** (1993), 115–118.

[15] V. I. Gurarij, M. I. Kadets and V. I. Macaev, *On the distance between isomorphic L_p spaces of finite dimension*, Matematicheskii Sbornik **70** (1966), 481–489.

[16] V. I. Gurarij and M. I Kadets, *On permutations of biorthogonal decompositions*, Istituto Lombardo A **125** (1991), 77–88.

[17] M. I. Kadets and A. Pełczyński, *Basic sequences, biorthogonal systems and norming sets in Banach and Frechet spaces*, Studia Mathematica **25** (1965), 297–323.

[18] M. I. Kadets, *Non-linear operatorial bases in a Banach space*, Teoriya Funktsii Funktsional Anal. i ikh Prilozheniya **2** (1966), 128–130 (Russian).

[19] V. M. Kadets, *Bases with individual brackets and bases with individual permutations*, Teoriya Funktsii Funktsional Anal. i ikh Prilozheniya **49** (1988), 43–51 (Russian).

[20] V. M. Kadets, A. N. Plichko and M. M. Popov, *Complete minimal systems of a certain type in Banach spaces*, Soviet Mathematics **32** (1988), 39–48.

[21] H. Konig and N. Tomczak-Jaegermann, *Norms of minimal projections*, Journal of Functional Analysis **2** (1994), 253–280.

[22] M. G. Krein, M. A. Krasnoselskii and D. P. Milman, *On defect numbers of linear operators in a Banach space and on some geometric problems*, Sbornik Trud. Inst. Matem. Akad. Nauk Ukr. SSR **11** (1948), 97–112.

[23] J. L. Krivine, *Sous espaces de dimension finis des spaces de Banach reticules*, Annals of Mathematics **104** (1976), 1–29.

[24] J. Lindenstrauss, *Decomposition of Banach spaces*, Indiana University Mathematics Journal **20** (1971), 917–919.

[25] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer, Berlin, 1977.

[26] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer, Berlin, 1979.

[27] B. Maurey and G. Pisier, *Séries de variables aléatories vectorielles indépéndantes et propriétés géometriques des espaces de Banach*, Studia Mathematica **58** (1976), 45–90.

[28] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Springer, Berlin, 1986.

[29] L. Nachbin, *A theorem of the Hahn–Banach type for linear transformations*, Transactions of the American Mathematical Society **68** (1950), 26–46.

[30] S. Negrepontis, *Banach spaces and topology*, in *Handbook of Set-Theoretic Topology* (K. Kunen and J. E. Vaughan, eds.), Elsevier Publisher B.V., Amsterdam, 1984.

[31] M. I. Ostrovskii, *On norming Markushevich bases*, Matematychni Studii (L'viv) **5** (1995), 39–42.

[32] R. I. Ovsepian and A. Pełczyński, *On the existence of a fundamental total and bounded biorthogonal sequence in every separable Banach space, and related constructions of uniformly bounded orthonormal systems in L^2*, Studia Mathematica **54** (1975), 149–159.

[33] A. Pełczyński, *All separable Banach spaces admit for every $\varepsilon > 0$ fundamental total and bounded by $1+\varepsilon$ biorthogonal sequences*, Studia Mathematica **55** (1976), 295–304.

[34] G. Pisier, *Holomorphic semi-groups and the geometry of Banach spaces*, Annals of Mathematics **115** (1982), 375–392.

[35] G. Pisier, *Counterexample to a conjecture of Grothendieck*, Acta Mathematica **151** (1983), 181–208.

[36] A. N. Plichko, *M-bases in separable and reflexive spaces*, Ukrainian Mathematical Journal **29** (1977), 681–685.

[37] A. N. Plichko, *The existence of a bounded M-basis in a WCG-space*, Teoriya Funktsii Funktsional Anal. i ikh Prilozeniya **32** (1979), 61–69.

[38] A. N. Plichko, *The existence of a bounded total biorthogonal system in a Banach space*, Teoriya Funktsii Funcktsional Anal. i ikh Prilozeniya **33** (1980), 111–118.

[39] A. N. Plichko, *A Banach space without a fundamental biorthogonal system*, Soviet Mathematics Doklady **22** (1980), 450–453.

[40] A. N. Plichko, *Fundamental biorthogonal systems and projections bases in Banach spaces*, Matematicheskie Zametki **33** (1983), 473–476 (Russian).

[41] A. N. Plichko, *Projection decompositions, Markushevich bases and equivalent norms*, Matematicheskie Zametki **34** (1983), 719–726 (Russian).

[42] A. N. Plichko, *Bases and complements in nonseparable Banach spaces*, Sibirskii Matematicheskii Zhurnal **25** (1984), 155–162 (Russian).

[43] A. N. Plichko, *On bounded biorthogonal systems in some function spaces*, Studia Mathematica **84** (1986), 25–37.

[44] S. Z. Revesz, *Rearrangement of Fourier series*, Journal of Approximation Theory **60** (1990), 101–121.

[45] I. Singer, *Bases in Banach Spaces I*, Springer, Berlin, 1970.

[46] I. Singer, *On biorthogonal systems and total sequences of functionals*, Mathematische Annalen **193** (1971), 183–188.

[47] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer, Berlin, 1971.

[48] I. Singer, *On the extension of basic sequences to bases*, Bulletin of the American Mathematical Society **80** (1974), 771–772.

[49] I. Singer, *Bases in Banach Spaces II*, Springer, Berlin, 1981.

[50] A. Sobczyk, *Projection of the space m on its subspace c_0*, Bulletin of the American Mathematical Society **47** (1941), 938–947.

[51] S. J. Szarek, *A Banach space without a basis which has the bounded approximation property*, Acta Mathematica **151** (1983), 153–179.

[52] P. Terenzi, *Extension of uniformly minimal M-basic sequences in Banach spaces*, Journal of the London Mathematical Society **27** (1983), 500–506.

[53] P. Terenzi, *Representation of the space spanned by a sequence in Banach spaces*, Archives of Mathematics **43** (1984), 448–459.

[54] P. Terenzi, *On the theory of fundamental bounded biorthogonal systems in Banach spaces*, Transactions of the American Mathematical Society (299) **2** (1987), 497–511.

[55] P. Terenzi, *On the basis problem*, Istituto Lombard, Rendiconti Scienze Matematiche e Applicazioni (127) **2** (1993), 169–226.

[56] P. Terenzi, *Basic sequences and associated sequences of functionals*, Journal of the London Mathematical Society (2) **50** (1994), 187–198.

[57] P. Terenzi, *Every separable Banach space has a bounded strong norming biorthogonal sequence which is also a Steinitz basis*, Studia Mathematica **111** (1994), 207–222.

[58] M. Valdivia, *On basic sequences in Banach spaces*, Note Matematica **12** (1992), 245–258.

[59] W. A. Veech, *Short proof of Sobczyk's theorem*, Proceedings of the American Mathematical Society **28** (1971), 627–628.

[60] A. Wilanski, *Functional Analysis*, Blaisdell, Waltham, Mass., 1964.

[61] A. Wilanski, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York, 1978.

[62] P. Wojtaszczyk, *Every separable Banach space containing c_0 has an RUC system*, Longhorn Notes, University of Texas at Austin, 1986, pp. 37–39.

[63] M. Zippin, *On perfectly homogeneous bases in Banach spaces*, Israel Journal of Mathematics **4** (1966), 265–272.

[64] M. Zippin, *The separable extension problem*, Israel Journal of Mathematics **26** (1977), 372–387.

[65] M. Zippin, *The range of a projection of small norm in l_1^n*, Israel Journal of Mathematics **39** (1981), 349–358.